M. Garden

Well-Ordered Sets

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Contents

1 Introduction

The present unit is part of the walk *The Cardinality of Sets* consisting of the following units:

1. Finite Sets and their Cardinalities [Garden 2020f]
2. Well-Ordered Sets (this unit)
3. Ordinal Numbers [Garden 2020g]
4. Cardinal Numbers [Garden 2020h]
5. Cardinal Arithmetic [Garden 2020i]
6. The Axiomatics of von Neumann, Bernays and Gödel [Garden 2020j]

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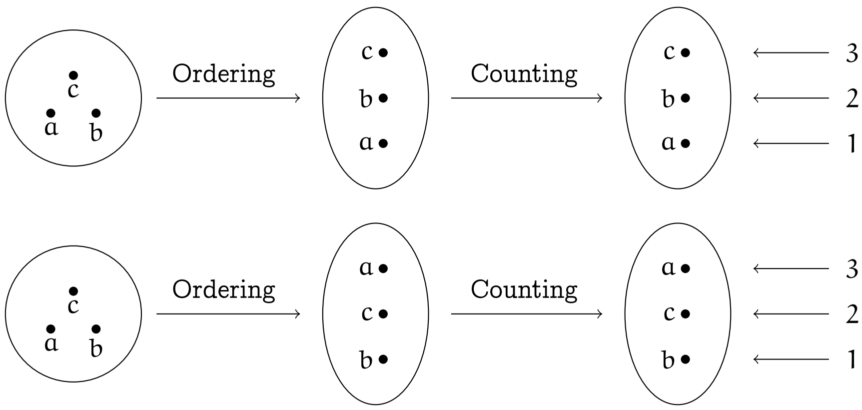
In 1874 Georg Cantor [Cantor 1874] discovered that there does not exist a bijective mapping between the set $\mathbb{N}_0$ of the natural numbers and the set $\mathbb{R}$ of the real numbers. This means that the cardinality of the set $\mathbb{R}$ is strictly greater than the cardinality of the set $\mathbb{N}_0$ (for more details see Unit *The Real Numbers* [Garden 2020k]).

This observation was the starting point for Cantor to develop a theory of the cardinality of (infinite) sets.

To follow his ideas, let us have a look at the finite situation: If we want to count the number of the elements of the set $A := \{a, b, c\}$, we construct a bijection $\alpha : \{1, 2, 3\} \to A$ from the set $\{1, 2, 3\}$ onto the set A. To do so we choose a first element of the set A to which we associate the number 1 and so on. In a certain sense we can say that we first order the set A and then define a bijection $\alpha : \{1, 2, 3\} \to A$. Here are two examples:

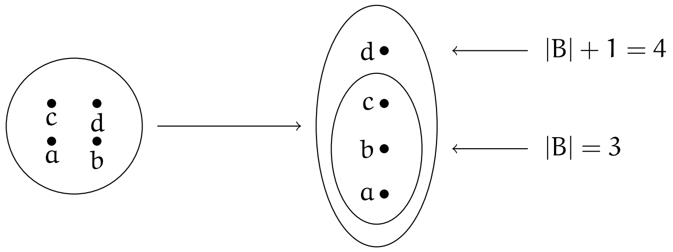


So the first observation of Cantor was that counting is closely related to ordering. His second observation was that counting is an inductive process: Suppose that we have a totally ordered set A of four elements, say $A = \{a, b, c, d\}$ with

$$a < b < c < d,$$

and suppose that we already have counted the number of elements of the "initial part" $B := \{a, b, c\}$ of the set A. Then we can proceed inductively by

$$|A| = |B| + 1.$$



To do so we must be able to identify the “next” element following the elements of the set B. In the finite case this is obvious. In the infinite case it is not. For example, if we consider the set $\mathbb{R}$ of the real numbers with the standard order $\leqslant$ and the subset $B := \{x \in \mathbb{R} \mid x \leqslant 1\}$, then there is no “next” element following the set B. In fact, if $z > 1$, then the number $\frac{1+z}{2}$ is a real number between 1 and z. In other words, the standard order $\leqslant$ on the set $\mathbb{R}$ of the real numbers does not fit to the idea of counting sketched above.

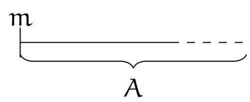
Cantor’s idea is to endow the set $\mathbb{R}$ and even every set A with a more suitable order. He calls an oder $\leqslant$ on a set A which allows the counting process described above a *well-ordering*, and it is time to give a formal definition:

Let us start with an arbitrary set A. As explained above Cantor wants to establish an (infinite) inductive counting process. The first requirement is to have a total order on the set A:

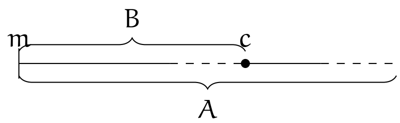
(W1) The set A is endowed with a total order $\leqslant$.

The second requirement is that Cantor wants to start the counting process with a first element:

(W2) The totally ordered set $A = (A, \leqslant)$ has a minimal element m.



Assuming that the cardinality of an “initial part” B of the set A has been established we need a “next” element c.



Let us first define what an *initial* part is and what a *next* element is:

A subset B of a totally ordered set $A = (A, \leqslant)$ is called an **initial part** of the set A if the following condition is fulfilled:

Let b be an element of the set B, and let x be an element of the set A such that $x \leqslant b$. Then the element x is contained in the set B.

Let us consider the following example: Let

$$W := \mathbb{N}_0 \uplus \{w_0\} \uplus \{w_1\}$$

with the order

$$0 < 1 \ldots < n < n+1 < \ldots < w_0 < w_1.$$

The sets $\emptyset$, $\{0, 1, \ldots, n\}$ $(n \in \mathbb{N}_0)$, $\mathbb{N}_0$, $\mathbb{N}_0 \cup \{w_0\}$ and $W = \mathbb{N}_0 \cup \{w_0, w_1\}$ are the initial parts of the set W.

Let A be a totally ordered set, let B be an initial part of the set A, and let c be an element of the set A. The element c is called the **successor of the set** B if the following conditions are fulfilled:

(i) We have

$$x < c \text{ for all } x \in B.$$

(ii) Suppose that x is an element of the set A with $x < c$. Then the element x is contained in the set set B.

One can also say that there is no element "between" the set B and the element c. We are now able to state our last requirement concerning a well-ordered set:

(W3) Let B be an initial part of the set A such that $B \neq A$. Then the set B has a successor.

In fact, Condition (W2) is Condition (W3) with $B = \emptyset$ and can therefore be omitted.

Let us look again at the set $W := \mathbb{N}_0 \cup \{w_0, w_1\}$ defined above. The successors of the initial parts $B \neq W$ are as follows:

Initial Part	Successor
$\emptyset$	0
$\{0, 1, \ldots, n\}$	$n + 1$
$\mathbb{N}_0$	w_0
$\mathbb{N}_0 \cup \{w_0\}$	w_1

Summarizing what has been said above we are now able to give a formal definition of a well-ordered set: An ordered set $A = (A, \leqslant)$ is called **well-ordered** if the following conditions are fulfilled:

(W1) The set $A = (A, \leqslant)$ is totally ordered.

(W2) The set $A = (A, \leqslant)$ has a minimal element.

(W3) Let B be an initial part of the set A such that $B \neq A$. Then the set B has a successor.

In this case the order $\leqslant$ is called a **well-ordering** of the set A.

As we have seen above the set $W := \mathbb{N}_0 \cup \{w_0, w_1\}$ with the order

$$0 < 1 \ldots < n < n + 1 < \ldots < w_0 < w_1.$$

is a well-ordered set.

Cantor was convinced that every set A can be well-ordered, that is, that for each set A there exists an order $\leqslant$ on the set A such that the pair $(A, \leqslant)$ is well-ordered. One can say that Cantor formulated the axiom that every set can be well-ordered.

In 1904 Ernst Zermelo [Zermelo 1904] was able to deduce the existence of well-orderings for every set A from the axiom of choice. As a consequence the axiom of Cantor became a theorem, the *well-ordering theorem*. On the other hand it is not so difficult to deduce the axiom of choice from the well-ordering axiom (see Theorem 13.1). So at the end Cantor was absolutely right to postulate the existence of well-orderings: The axiom of choice and the well-ordering theorem are equivalent: If one of the two is required as an axiom, then the other one can be proven.

Cantor also showed the following theorem about well-ordered sets: Let $A = (A, \leqslant)$ be a totally ordered set. Then the following conditions are equivalent:

(i) The pair $A = (A, \leqslant)$ is well-ordered.

(ii) Every non-empty subset of the set A has a minimal element.

Due to this equivalence nowadays a well-ordered set is generally defined as follows: A totally ordered set $A = (A, \leqslant)$ is called **well-ordered** if every non-empty subset of the set A has a minimal element (see Definition 5.1).

In the present unit we will explain the important properties of well-ordered sets, and we will present two different proofs of Zermelo's fundamental result that every set can be well-ordered.

The rest of the story, namely how to use well-orderings for establishing the cardinality of infinite sets will be explained in the two subsequent Units *Ordinal Numbers* [Garden 2020g] and *Cardinal Numbers* [Garden 2020h].

The material in this unit is organized as follows:

Ordered Sets (see Section 2):

In this section we will recall some facts about ordered sets.

Successors and Predecessors (see Section 3):

The number 5 follows the number 4 in the set $\mathbb{N}_0$ of the natural numbers with the standard order. We also say that the number 5 is the successor of the number 4. We generalize this term to arbitrary ordered sets. For an ordered set $A = (A, \leqslant)$ and two elements x and y of the set A we say that the element y is a **successor** of the element x if we have $x < y$ and if there is no element of the set A between the elements x and y (see Definition 3.1).

Successors of Sets (see Section 4):

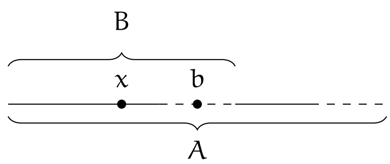
More interesting is the term of a successor of a set. The definition is quite similar to the definition of a successor of an element: If a is an element and if B is a subset of an ordered set $A = (A, \leqslant)$, then the element a is called a **successor of the set** B if

$$x < a \text{ for all } x \in B$$

and if there is no element between the set B and the element a (see Definition 4.1).

The most important result of this section is Theorem 4.9: As we have explained above, the term *initial part* of a set is defined as follows: Let $A = (A, \leqslant)$ be an ordered set. A subset B of the set A is called an **initial part** of the set A if the following condition is fulfilled:

If b is an element of the set B and if x is an element of the set A such that $x < b$, then the element x is contained in the set B.



Theorem 4.9 says that an initial part B of an ordered set A is an initial segment

$$A_a = \{x \in A \mid x < a\}$$

for an element a of the set A if and only if the element a is a successor of the set B.

Well-Ordering (see Section 5):

In this section we will introduce a well-ordered set A as a totally ordered set $A = (A, \leqslant)$ with the property that every non-empty subset of the set A has a minimal element (see Definition 5.1).

We will see that a subset of a well-ordered set is well-ordered (Proposition 5.6), that the set $\mathbb{N}_0$ of the natural numbers is well-ordered (Theorem 5.4) and that a one point extension of a well-ordered set is well-ordered (Theorem 5.8).

Characterizations of Well-Ordered Sets (see Section 6):

In this section we will bring together the terms well-ordered set, successor and initial part: Theorem 6.1 says that a totally ordered set A is well-ordered if and only if any initial part of the set A is an initial segment, that is, if it has a successor.

This theorem is the basis for the counting approach of Cantor: If the cardinality of an initial part B of a well-ordered set A has been determined, then we can go one step further by adding the successor of the set B. We will explain this in more detail in Unit *Ordinal Numbers* [Garden 2020g].

Initial Segments and Well-Ordered Sets (see Section 7):

Initial segments play a fundamental role in the study of well-ordered sets. In this section we explain the most important results:

Let a be an element of a well-ordered set A. Then either the element a is the maximum of the set A or the set $\{x \in A \mid x \leqslant a\}$ has a success or b. In this case we have

$$\{x \in A \mid x \leqslant a\} = \{x \in A \mid x < b\}.$$

One can say that an initial segment A_a is either the "last" initial segment (in this case the element a is the maximal element of the set A) or there is a "next" initial segment A_b (in this case the element b is the successor of the initial segment A_a (see Theorem 7.2). For example, the initial segment $\{x \in \mathbb{N}_0 \mid x < n\}$ is followed by the initial segment $\{x \in \mathbb{N}_0 \mid x < n+1\}$.

On the other hand, an initial segment does not have to have a "previous" initial segment. Firstly, the empty set is always the "first" initial segment without a "previous" initial segment. Secondly, if b is an element of a well-ordered set, then the element b has either a predecessor a (in this case we have $\{x \in A \mid x \leqslant a\} = \{x \in A \mid x < b\}$; the initial segment A_a is "previous" to the initial segment A_b) or the element b has no predecessor. In this case we have

$$A_b = \bigcup_{z<b} A_z.$$

See Corollary 7.7. For example, let $A := \mathbb{N}_0 \cup \{\mathbb{N}_0\}$. The element 6 has the predecessor 5, the initial segment $\{x \in A \mid x < 5\}$ is previous to the initial segment $\{x \in A \mid x < 6\}$. The element $\mathbb{N}_0$ has no predecessor, but we have

$$\mathbb{N}_0 = A_{\mathbb{N}_0} = \bigcup_{z<\mathbb{N}_0} A_z = \bigcup_{n\in\mathbb{N}_0} A_n = \bigcup_{n\in\mathbb{N}_0} \{x \in \mathbb{N}_0 \mid x < n\}.$$

Transfinite Induction (see Section 8):

Transfinite Induction is a generalization of the principle of induction from the natural numbers to well-ordered set. In the principle of induction we conclude from a natural number n to the next natural number $n+1$. In the principle of transfinite induction we conclude from the elements of an initial segment A_a to the next element (successor) a. The details are explained in Theorem 8.1.

The Well-Ordering Theorem (see Section 9):

The Well-Ordering Theorem (Theorem 9.7) is one of the central results of this unit. It says that every set can be endowed with a well-ordering. The proof will be based on the Lemma of Zorn.

Final Segments in Well-Ordered Sets (see Section 10):

This section prepares the second proof of the well-ordering theorem presented in Section 11. This proof is based on the final segments $T_a := \{x \in A \mid x \geqslant a\}$ of an ordered set. The main result is an abstract description of the set of the final segments of a well-ordered set:

Let $\mathcal{T} := \{T_a \mid a \in A\} \cup \{\emptyset\}$ be the set of the final segments of a well-ordered set A together with the empty set, and let $\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ be defined by $\alpha(X) := \min(X)$ where $\min(X)$ denotes the minimal element of the set X. Note that the function $\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ is a choice function.

Then the set $\mathcal{T}$ has the following properties (see Proposition 10.6):

(i) We have $\alpha(T_a) = a$ for all elements a of the set A.

(ii) The sets $\emptyset$ and A are elements of the set $\mathcal{T}$.

(iii) For each element a of the set A the set $T_a \setminus \{\alpha(T_a)\} = T_a \setminus \{a\}$ is an element of the set $\mathcal{T}$, that is, we have

$$X \setminus \{\alpha(X)\} \in \mathcal{T} \text{ for all } X \in \mathcal{T}, X \neq \emptyset.$$

(iv) If $(X_i)_{i \in I}$ is a family of sets of the set $\mathcal{T}$ for some index set I, then we have

$$\bigcap_{i \in I} X_i \in \mathcal{T}.$$

(v) If $\mathcal{A}$ is further set with the properties (ii) (that is, $\emptyset \in \mathcal{A}$ and $a \in \mathcal{A}$), (iii) (that is, $(X \setminus \{\alpha(X)\}) \in \mathcal{A}$ for all $X \in \mathcal{A}$) and (iv) (that is, $\bigcap_{i \in I} X_i \in \mathcal{A}$ for all families $(X_i)_{i \in I}$ of sets of the set $\mathcal{A}$), then the set $\mathcal{T}$ is a subset of the set $\mathcal{A}$. In other words, the set $\mathcal{T}$ is minimal with respect to the Properties (ii), (iii) and (iv).

A Second Proof of the Well-Ordering Theorem (see Section 11):

The second proof of the well-ordering theorem will be based on the axiom of choice. In view of the results in Section 10 we define a θ-chain as follows:

Given a non-empty set A and an arbitrary choice function $\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ a subset $\mathcal{A}$ of the power set $\mathcal{P}(A)$ is called a **θ-chain with respect to the pair** (A, α) if it fulfills Conditions (ii), (iii) and (iv) described above (see Definition 10.3).

In Section 11 we will show that for a set A and a choice function $\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ there exists a minimal θ-chain $\mathcal{T}$ with respect to the pair (A, α) (Proposition 11.1). We will use the properties of this set $\mathcal{T}$ to define an order on the set A and to show that this order is a well-ordering on the set A. For details see the proof of Theorem 11.3.

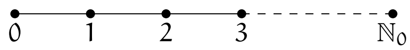
Isomorphisms of Well-Ordered Sets (see Section 12):

In this section we will show the following powerful result about well-ordered sets (see Theorem 12.10): Given two well-ordered sets A and B, then one of the following cases occurs:

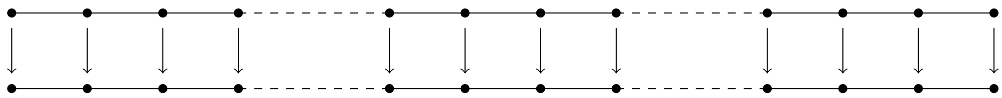
(i) The sets A and B are isomorphic ordered sets.

(ii) The set A is isomorphic to an initial segment of the set B, or the set B is isomorphic to an initial segment of the set A.

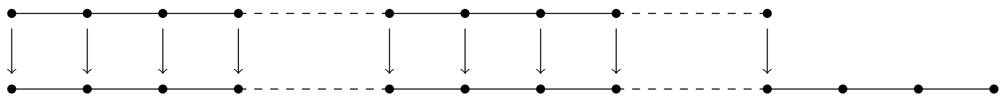
Intuitively we can imagine the two well-ordered sets as two pearl necklaces where each pearl has a successor, but some pearls do not have a predecessor. For example, the well-ordered set $\mathbb{N}_0 \cup \{\mathbb{N}_0\}$ looks as follows



Theorem 12.10 says that two well-ordered sets A and B are either "pearl necklaces" of the same "length" and "form"



or one of the two sets (pearl necklaces) looks like the beginning of the other one.



The Equivalence of the Axiom of Choice, the Lemma of Zorn and the Well-Ordering Theorem (see Section 13):

In the last section we will show that the axiom of choice, the Lemma of Zorn and the well-ordering theorem are all equivalent. This means that one could replace the axiom of choice in the list of the axioms of Zermelo and Fraenkel by the Lemma of Zorn or, equivalently, by the well-ordering theorem (see Theorem 13.1).

2 Ordered Sets

Ordered Sets are explained in detail in Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. We will recall in this section those properties of ordered sets needed in the present unit:

Ordered Sets:

2.1 Definition. Let A be a set.

(a) A relation $\leqslant$ on the set A is called an **order** or, equivalently, a **partial order on the set** A if it fulfills the following conditions:

(i) The relation $\leqslant$ is **reflexive**, that is, $x \leqslant x$ for all elements x of the set A.

(ii) The relation $\leqslant$ is **antisymmetric**, that is, $x \leqslant y$ and $y \leqslant x$ imply $x = y$ for all elements x and y of the set A.

(iii) The relation $\leqslant$ is **transitive**, that is, $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$ for all elements x, y and z of the set A.

(b) A set A with a (partial) order $\leqslant$ is called an **ordered set** or, equivalently, a **partially ordered set** and is often denoted by $A = (A, \leqslant)$.

(c) A (partial) order $\leqslant$ on a set A is called a **total order** if we have $x \leqslant y$ or $y \leqslant x$ for all elements x and y of the set A. A set A with a total order $\leqslant$ is called a **totally ordered set**.

2.2 Definition. Let $A = (A, \leqslant_A)$ be an ordered set, let B be a subset of the set A, and let $\leqslant_B$ be the order on the set B such that

$$x \leqslant_B y \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in B.$$

The order $\leqslant_B$ is called **the order on the set B induced by the order $\leqslant_A$**.

Often we write $A = (A, \leqslant)$ and $B = (B, \leqslant)$ without explicitly distinguishing between the order $\leqslant_A$ on the set A and the induced order $\leqslant_B$ on the set B.

Isomorphisms of Ordered Sets:

2.3 Definition. Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \longrightarrow B$ be a mapping from the set A into the set B.

(a) The mapping $\alpha : A \longrightarrow B$ is called a **homomorphism of the set A into the set B** if $x \leqslant_A y$ implies $\alpha(x) \leqslant_B \alpha(y)$ for all elements x and y of the set A.

(b) The mapping $\alpha : A \longrightarrow B$ is called an **isomorphism of the set A onto the set B** if and only if the mapping $\alpha : A \longrightarrow B$ is bijective and if we have

$$\alpha(x) \leqslant_B \alpha(y) \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in A.$$

(c) If $\alpha : A \longrightarrow B$ is an isomorphism from the set A onto the set B, then the sets A and B are called **isomorphic**. In this case we write $A \cong B$.

2.4 Proposition. *Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \longrightarrow B$ be an isomorphism from the set A onto the set B.*

Then the function $\alpha^{-1} : B \longrightarrow A$ is an isomorphism from the set B onto the set A.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

2.5 Proposition. *Let* $(A, \leqslant_A)$, $(B, \leqslant_B)$ *and* $(C, \leqslant_C)$ *be three ordered sets.*

(a) Suppose that there exist two isomorphisms $\alpha : A \to B$ *and* $\beta : B \to C$ *from the set* A *onto the set* B *and from the set* B *onto the set* C*, respectively.*

Then the composite $\gamma := \beta \circ \alpha : A \to C$ *is an isomorphism from the set* A *onto the set* C.

(b) If $A \cong B$ *and* $B \cong C$*, then we have* $A \cong C$.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

Initial and Final Segments:

2.6 Definition. Let $(A, \leqslant)$ be an ordered set, and let a be an element of the set A.

(a) The set

$$A_a := \{x \in A \mid x < a\}$$

is called the **initial segment of the set** A **with respect to the element** a.

(b) The set

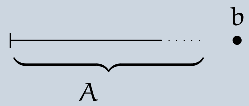
$$T_a := \{x \in A \mid x \geqslant a\}$$

is called the **final segment of the set** A **with respect to the element** a.

One Point Extensions:

2.7 Definition. Let $A = (A, \leqslant_A)$ be a totally ordered set, and let b be an element not contained in the set A. Set $B := A \,\dot\cup\, \{b\}$. We define an order $\leqslant_B$ on the set B as follows:

For each two elements x and y of the set A, set $x \leqslant_B y$ if and only if $x \leqslant_A y$. For each element x of the set B, set $x \leqslant_B b$.



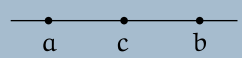
The pair $(B, \leqslant_B)$ is called the **one point extension** of the pair $(A, \leqslant_A)$.

3 Successors and Predecessors

Definition of Successors and Predecessors:

3.1 Definition. Let $A = (A, \leqslant)$ be a totally ordered set, and let a and b be two elements of the set A.

(a) An element c of the set A is called an **element between the elements** a **and** b if we have



$$a < c < b \text{ or } b < c < a.$$

(b) The element b is called the **successor of the element** a if we have $a < b$ and if there is no element of the set A between the elements a and b.

(c) The element a is called the **predecessor of the element** b if we have $a < b$ and if there is no element of the set A between the elements a and b, that is, if the element b is the successor of the element a.

French / German. Successor = Successeur = Nachfolger; Predecessor = Prédécesseur = Vorgänger.

3.2 Examples. (a) The number 3 is between the numbers 2 and 5.

(b) Let n be a natural number. Then the number $n+1$ is the successor of the number n, and the number n is the predecessor of the number $n+1$.

(c) Let $A = (\mathbb{Q}, \leqslant)$ or $A = (\mathbb{R}, \leqslant)$ be the set of the rational or the real numbers with the standard order. No element of the set A has a successor or a predecessor.

(d) Let $A = (A, \leqslant)$ be a totally ordered set. If the set A has a minimum a, then the element a has no predecessor. If the set A has a maximum b, then the element b has no successor.

3.3 Remark. Note that we have used the word successor twice: In Unit *Successor Sets and the Axioms of Peano* [Garden 2020d] the set $A := A \cup \{A\}$ is called the successor of the set A. For well-ordered sets, we have just introduced a second meaning for the word successor. It will be clear from the context how to understand the word successor.

Elementary Properties of Successors and Predecessors:

3.4 Proposition. *Let $A = (A, \leqslant)$ be a totally ordered set.*

(a) Each element of the set A has at most one successor.

(b) Each element of the set A has at most one predecessor.

Proof. (a) Let a be an element of the set A. Assume that the element a has two distinct successors b and b'. It follows that $a < b$ and $a < b'$. Since the pair $(A, \leqslant)$ is totally ordered, we have $b < b'$ or $b' < b$ implying that $a < b < b'$ or $a < b' < b$, in contradiction to the assumption that the elements b and b' are both successors of the element a.

(b) The proof is as in (a). □

4 Successors of Sets

Definition of the Successor of a Set:

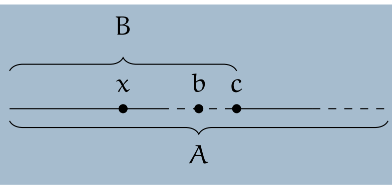
4.1 Definition. Let $A = (A, \leqslant)$ be a totally ordered set, and let B be a subset of the set A. An element c of the set A is called a **successor of the set** B if the following conditions are fulfilled:

(i) We have

$$x < c \text{ for all } x \in B.$$

(ii) If x is an element of the set A such that $x < c$, then there exists an element b of the set B with

$$x \leqslant b < c.$$



French / German. Successor of a set = Successeur d'un ensemble = Nachfolger einer Menge.

4.2 Examples. (a) Let $A := (\mathbb{R}, \leqslant)$ be the set of the real numbers with the standard order. The successor of the set $\{x \in \mathbb{R} \mid x < 1\}$ is the number 1.

(b) Let $A := (\mathbb{Q}, \leqslant)$ be the set of the rational numbers with the standard order. The set $\{x \in \mathbb{Q} \mid x < \sqrt{2}\}$ has no successor.

(c) Let $A := (\mathbb{N}, \leqslant)$ be the set of the natural numbers with the standard order. The successor of the set $\{1, 3, 5\}$ is the number 6.

Elementary Properties of the Successor of a Set:

4.3 Proposition. *Let $A = (A, \leqslant)$ be a totally ordered set, and let B be a subset of the set A. Then there exists at most one successor of the set B.*

Proof. Assume that there exist two successors c and c'. W.l.o.g. suppose that $c' < c$. By definition of a successor (Definition 4.1), there exists an element b of the set B such that $c' \leqslant b$, a contradiction. □

4.4 Proposition. *Let $A = (A, \leqslant)$ be a totally ordered set. Then the empty set $\emptyset$ has a successor if and only if the set A has a minimum.*

Proof. Step 1. *Suppose that the empty set has a successor c. Then the element c is the minimum of the set A:*

Assume that the element c is not the minimum of the set A. Then there exists an element x of the set A such that $x < c$. Since the element c is the successor of the set $B := \emptyset$, there exists an element b of the set B such that $x \leqslant b < c$, in contradiction to $B = \emptyset$.

Step 2. *Suppose that the set A has a minimum c. Then the element c is the successor of the empty set:*

Assume that the element c is not the successor of the empty set. Then there exists an element x of the set A such that $x < c$ and such that

$$b < x \text{ for all } b \in B,$$

in contradiction to the assumption that the element c is the minimal element of the set A. □

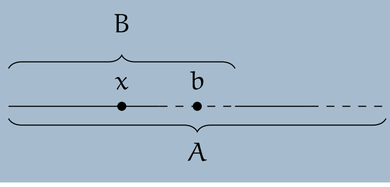
4.5 Proposition. *Let $A = (A, \leqslant)$ be a totally ordered set, and let a be an element of the set A. Then the element a is the successor of the initial segment*

$$A_a := \{x \in A \mid x < a\}.$$

Proof. The proof is obvious. □

4.6 Definition. Let A be a totally ordered set, and let B be a subset of the set A. The set B is called an **initial part of the set** A if the following condition is fulfilled:

If b is an element of the set B and if x is an element of the set A such that $x < b$, then the element x is contained in the set B.



4.7 Proposition. *Let $A = (A, \leqslant)$ be a totally ordered set, and let a be an element of the set A. Then the sets*

$$\{x \in A \mid x < a\} \text{ and } \{x \in A \mid x \leqslant a\}$$

are initial parts of the set A.

Proof. The proof is obvious. □

4.8 Example. Let $A := (\mathbb{Q}, \leqslant)$ be the set of the rational numbers with the standard order, and let

$$B := \{x \in \mathbb{Q} \mid x < \sqrt{2}\}.$$

Then the set B is an initial part of the set $A = \mathbb{Q}$, but there is no element a of the set $A = \mathbb{Q}$ such that

$$A = \{x \in A \mid x < a\} \text{ or } A = \{x \in A \mid x \leqslant a\}.$$

4.9 Theorem. *Let $A = (A, \leqslant)$ be a totally ordered set, let B be an initial part of the set A, and let c be an element of the set A. Then the following conditions are equivalent:*

(i) The element c is a successor of the set B.

(ii) We have

$$B = A_c := \{x \in A \mid x < c\}.$$

Proof. (i) $\Rightarrow$ (ii): By definition of a successor (Definition 4.1), we have

$$x < c \text{ for all } x \in B,$$

that is

$$B \subseteq A_c = \{x \in A \mid x < c\}.$$

Let x be an element of the set A such that $x < c$. By definition of a successor (Definition 4.1), there exists an element b of the set B such that $x \leqslant b$. By definition of an initial part (Definition 4.6), the element x is contained in the set B. It follows that

$$A_c = \{x \in A \mid x < c\} \subseteq B.$$

(ii) $\Rightarrow$ (i): The proof follows from Proposition 4.5. □

4.10 Corollary. *Let $A = (A, \leqslant)$ be a totally ordered set, and let a and b be two elements of the set A. Then the following statements are equivalent:*

(i) The element b is a successor of the element a.

(ii) The element a is a predecessor of the element b.

(iii) The element b is a successor of the set

$$\bar{A}_a := \{x \in A \mid x \leqslant a\}.$$

(iv) We have

$$\bar{A}_a = A_b.$$

Proof. The proof is obvious. □

5 Well-Ordering

Definition of Well-Ordered Sets:

5.1 Definition. Let $A = (A, \leqslant)$ be a totally ordered set.

(a) The pair $A = (A, \leqslant)$ is called **well-ordered** if every non-empty subset B of the set A has a minimal element, that is, if for every subset B of the set A there exists an element b of the set B such that

$$x \geqslant b \text{ for all } x \in B.$$

(b) If the pair $A = (A, \leqslant)$ is well-ordered, then the order $\leqslant$ is called a **well-ordering**.

French / German. Well-ordered set = Ensemble bien ordonné = Wohlgeordnete Menge.

Elementary Properties of Well-Ordered Sets:

5.2 Proposition. *The pair $(\emptyset, \emptyset)$ is well-ordered.*

Proof. Note that the pair $(\emptyset, \emptyset)$ is an ordered set (for more details see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]). Since there are no non-empty subsets of the empty set, the pair $(\emptyset, \emptyset)$ is obviously well-ordered. □

5.3 Theorem. *Let $A = (A, \leqslant)$ be a finite totally ordered set. Then the pair $(A, \leqslant)$ is well-ordered.*

Proof. We have shown in Unit *Finite Sets and their Cardinalities* [Garden 2020f] that every totally ordered finite set has a minimal element. The assertion follows. □

5.4 Theorem. *The pair $(\mathbb{N}_0, \leqslant)$ is well-ordered (where $\leqslant$ denotes the standard order).*

Proof. Let A be a non-empty subset of the set $\mathbb{N}_0$, and let

$$M := \{x \in \mathbb{N}_0 \mid x \leqslant a \text{ for all } a \in A\}.$$

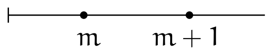
Step 1. *The set M contains the number 0:*

The assertion is obvious since $0 \leqslant n$ for all natural numbers n.

Step 2. *We have $M \neq \mathbb{N}_0$:*

Since $\emptyset \neq A$, there exists a number a of the set A. Since $a < a+1$, the number $a+1$ is not contained in the set M.

Step 3. *There exists a number m of the set M such that the number $m+1$ is not contained in the set M.*



Otherwise, we have

$$0 \in M \text{ (Step 1) and } x+1 \in M \text{ for all } x \in M.$$

By induction, we get $M = \mathbb{N}_0$, in contradiction to Step 2.

Step 4. *The number m of Step 3 is the minimal element of the set A:*

Since the number m is contained in the set M, we have $m \leqslant x$ for all numbers x of the set A.

Assume that the number m is not contained in the set A. Then we have $m \leqslant x$ and $m \neq x$ for all numbers x of the set A, that is, $m < x$ for all numbers x of the set A.

It follows that $m+1 \leqslant x$ for all numbers x of the set A, that is, the number $m+1$ is contained in the set M, in contradiction to Step 3. □

5.5 Remark. The assertion of Theorem 5.4 should not be misunderstood: We claim that the pair $(\mathbb{N}_0, \leqslant)$ with the standard order $\leqslant$ is a well-ordered set. We do not claim that *every* order on the set $\mathbb{N}_0$ is a well-ordering. For example, the order $\geqslant$ is no well-ordering on the set $\mathbb{N}_0$: Set

$$m \leqslant' n \text{ if and only if } m \geqslant n \text{ for all } n, m \in \mathbb{N}_0.$$

Then the pair $(\mathbb{N}_0, \leqslant')$ has no minimal element. Hence, the order $\leqslant'$ is no well-ordering.

5.6 Proposition. *Let $(A, \leqslant)$ be a well-ordered set, and let B be a subset of the set A with the induced order. Then the pair $(B, \leqslant)$ is well-ordered.*

Proof. Let X be a non-empty subset of the set B. Since the set B is a subset of the set A, the set X is a subset of the set A. Since the pair $(A, \leqslant)$ is well-ordered, the set X has a minimal element. It follows that the pair $(B, \leqslant)$ is well-ordered. □

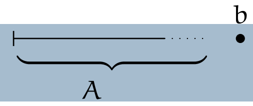
5.7 Theorem. *(a) The pair $(\mathbb{N}, \leqslant)$ is well-ordered (where $\leqslant$ denotes the standard order).*
(b) Let n be a natural number. Then the pair $(n, \leqslant)$ is well-ordered.

Proof. (a) Since the set $\mathbb{N}$ is a subset of the set $\mathbb{N}_0$, the assertion follows from Theorem 5.4 and Proposition 5.6.

(b) By Theorem 5.3, every finite totally ordered set is well-ordered. □

5.8 Theorem. *Let $A = (A, \leqslant_A)$ be a totally ordered set, and let $B = A \cup \{b\}$ be the one point extension of the set A (Definition 2.7).*

If the pair $(A, \leqslant)$ is well-ordered, then the pair $(B, \leqslant)$ is well-ordered, too.



Proof. Let X be a non-empty subset of the set B.

Case 1. *Suppose that* $X \cap A \neq \emptyset$.

Then there exists a minimal element a of the set $X \cap A$ with respect to the pair $(A, \leqslant_A)$. Since $a \leqslant b$, it follows that the element a is also a minimal element of the set X with respect to the pair $(B, \leqslant_B)$.

Case 2. *Suppose that* $X \cap A = \emptyset$.

Then we have $X = \{b\}$, and the element b is the minimal element of the set X with respect to the pair $(B, \leqslant_B)$. □

5.9 Proposition. *Let $W := \mathbb{N}_0 \cup \{\mathbb{N}_0\}$. Define the order $\leqslant_W$ on the set W as follows: Set*

$$m \leqslant_W n \text{ if and only if } m \leqslant n \text{ for all } m, n \in \mathbb{N}_0 \text{ and } n \leqslant_W \mathbb{N}_0 \text{ for all } n \in \mathbb{N}_0.$$

Then the pair $(W, \leqslant_W)$ is well-ordered.

Proof. By Theorem 5.4, the pair $(\mathbb{N}_0, \leqslant)$ is well-ordered. It follows from Theorem 5.8 that the pair $(W, \leqslant_W)$ is well-ordered. □

Initial Segments:

5.10 Proposition. *Let $(A, \leqslant)$ be a well-ordered set, and let a be an element of the set A.*

(a) The initial segment $A_a := \{x \in A \mid x < a\}$ of the set A with respect to the element a is well-ordered.

(b) The set $\bar{A}_a := \{x \in A \mid x \leqslant a\}$ is well-ordered.

Proof. The assertions follow from Proposition 5.6. □

Historical Notes:

Well-ordered sets have been introduced by Georg Cantor:

Unter einer **wohlgeordneten Menge** *ist jede wohldefinierte Menge zu verstehen, bei welcher die Elemente durch eine bestimmt vorgegebene Sukzession miteinander verbunden sind, welcher gemäß es ein erstes Element der Menge gibt und sowohl auf jedes einzelne Element (falls es nicht das letzte in der Suzession ist) ein bestimmtes anderes folgt, wie auch zu jeder beliebigen endlichen oder unendlichen Menge von Elementen ein bestimmtes Element gehört, welches das ihnen allen nächstfolgende Element in der Sukzession ist (es sei denn, dass es ein ihnen allen in der Sukzession folgendes überhaupt nicht gibt).*

See [Cantor 1883, p. 548].

A **well-ordered set** *is to be understood as any well-defined set in which the elements are connected to one another by a certain predetermined succession, according to which* *there is a first element of the set and to each individual element (if it is not the last in the succession) a certain other follows, just as a certain element belongs to any arbitrary finite or infinite set of elements, which is the element next to them all in the succession (unless there is no element following them all in the succession).*

In a later article [Cantor 1897] Cantor summarizes and extends his results about well-ordered sets. In the following we will cite his results from that work even if he obtained some results perhaps a little bit earlier. His definition of well-ordered sets reads in [Cantor 1897] as follows:

Wohlgeordnet *nennen wir eine einfach geordnete Menge* F *[...], wenn ihre Elemente* f *von einem niedersten* f_1 *an in bestimmter Sukzession aufsteigen, so dass folgende zwei Bedingungen erfüllt sind:*

I. Es gibt in F *ein dem Range nach niederstes Element* f_1*.*

II. Ist F' *irgend eine Teilmenge von* F *und besitzt* F *ein oder mehrere Elemente höheren Ranges als alle Elemente von* F'*, so existiert ein Element* f' *von* F*, welches auf die Gesamtheit* F' *zunächst folgt, so dass keine Elemente in* F *vorkommen, die ihrem Range nach zwischen* F' *und* f' *fallen.*

See [Cantor 1897, p. 207].

We call a simply ordered set F **well-ordered** *[...] if its elements* f *rise in a certain succession from a lowest element* f_1 *on, so that the following two conditions are met:*

I. The set F *has a minimal element* f_1*.*

II. If F' *is any subset of* F *and if* F *has one or more elements that are bigger than all elements of* F'*, then there exists an element* f *of* F *which is bigger than all elements of* F'*, so that there are no elements in* F *between* F' *and* f*.*

Nowadays the standard definition of a well-ordered set is Definition 5.1. In Theorem 6.2 we will see that the original definition of Cantor is equivalent to our definition of a well-ordered set. In Theorem 6.2 we do not explicitly mention the existence of a minimal element because the successor of the empty set is the minimal element (if it exists).

6 Characterizations of Well-Ordered Sets

6.1 Theorem. *Let* $\mathbb{A} = (A, \leqslant)$ *be a totally ordered set. Then the following conditions are equivalent:*

(i) The pair $\mathbb{A} = (A, \leqslant)$ *is well-ordered.*

(ii) Let B *be an initial part of the set* A*, and let* $B \neq A$*. Then the set* B *is an initial segment of the set* A*.*

(iii) Let B *be an initial part of the set* A*, and let* $B \neq A$*. Then the set* B *has a successor.*

Proof. (i) $\Rightarrow$ (ii): Let B be an initial part of the set A, and let $B \neq A$. Since $B \neq A$, the set $C := A \setminus B$ is non-empty. Since the pair $(A, \leqslant)$ is well-ordered, the set C has a minimum c. Note that the element c is not contained in the set B.

We will apply Theorem 4.9:

We claim that $B = \{x \in A \mid x < c\}$:

For, let x be an element of the set B. Assume that $c \leqslant x$. Since the set B is an initial part of the set A, it follows that the element c is contained in the set B, a contradiction.

Conversely, suppose that x is an element of the set A with $x < c$. Since the element c is the minimum of the set $C = A \setminus B$, the element x is contained in the set B.

(ii) $\Rightarrow$ (iii): Suppose that there exists an element c of the set A such that the set B is the initial segment A_c. It follows from Proposition 4.5 that the element c is a successor of the set B.

(iii) $\Rightarrow$ (i): Let C be a non-empty subset of the set A. Let

$$B := \{x \in A \mid x < z \text{ for all } z \in C\}.$$

Obviously, the set B is an initial part of the set A, and we have $B \cap C = \emptyset$. Since the set C is non-empty, we have $B \neq A$. By Condition (iii), the initial part B has a successor c. By Theorem 4.9, we have

$$B = \{x \in A \mid x < c\}.$$

We claim that

$$c \leqslant z \text{ for all } z \in C:$$

For, assume that there exists an element c' of the set C such that $c' < c$. It follows that the element c' is contained in the set B, in contradiction to $B \cap C = \emptyset$. The element c is contained in the set C: Otherwise, we have $c < z$ for all elements z of the set C. It follows that the element c is contained i the set B, in contradiction to $B \cap C = \emptyset$. □

6.2 Theorem. *Let $A = (A, \leqslant)$ be a totally ordered set. Then the following conditions are equivalent:*

(i) The pair $A = (A, \leqslant)$ is well-ordered.

(ii) Let B be a subset of the set A such that there exists an element a of the set A with

$$x < a \textit{ for all } x \in B.$$

Then the set B has a successor.

Proof. (i) $\Rightarrow$ (ii): Let B be a subset of the set A such that there exists an element a of the set A with

$$x < a \text{ for all } x \in B.$$

Let

$$B' := \{x \in A \mid \exists\, b \in B \text{ s. t. } x \leqslant b\}.$$

Obviously, the set B' is an initial part of the set A. Since the element a is not contained in the set B', we have $B' \neq A$. By Theorem 6.1, the set B' has a successor c. It follows from Theorem 4.9 that we have

$$B' = \{x \in A \mid x < c\}.$$

We claim that the element c is a successor of the set B: Since the set B is a subset of the set B', we have

$$x < c \text{ for all } x \in B.$$

Let x be an element of the set A with $x < c$. It follows that the element x is contained in the set $B' = \{x \in A \mid x < c\}$, that is, there exists an element b of the set B such that $x \leqslant b < c$. Hence the element c is a successor of the set B.

(ii) $\Rightarrow$ (i): Let B be an initial part of the set A, and suppose that $B \neq A$. Then there exists an element a of the set $A \setminus B$. It follows that

$$x < a \text{ for all } x \in B.$$

By Condition (ii), the initial part B has a successor. It follows from Theorem 6.1 that the pair $(A, \leqslant)$ is well-ordered. □

Historical Notes:

Cantor already knew that his definition of a well-ordered set is equivalent to Definition 5.1:

A. *Jede Teilmenge F_1 einer wohlgeordneten Menge F hat ein niederstes Element.*

B. *Ist eine einfach geordnete Menge F so beschaffen, dass sowohl F wie auch jede ihrer Teilmengen ein niederstes Element haben, so ist F eine wohlgeordnete Menge.*

See [Cantor 1897, p. 208].

A. *Each subset F_1 of a well-ordered set F has a minimal element.*

B. *Suppose that a totally ordered set F has the property that F and that every subset of F has a minimal element. Then F is well-ordered.*

7 Initial Segments and Well-Ordered Sets

7.1 Theorem. *Let $A = (A, \leqslant)$ be a well-ordered set, and let B be an initial part of the set A. Then one of the following possibilities occurs:*

(i) We have $B = A$.

(ii) The set B is an initial segment of the set A, that is, there exists an element a of the set A such that

$$B = \{x \in A \mid x < a\}.$$

Proof. The assertion follows from Theorem 6.1. □

7.2 Theorem. *Let $A = (A, \leqslant)$ be a well-ordered set, let a be an element of the set A, and let*

$$\bar{A}_a := \{x \in A \mid x \leqslant a\}.$$

Then one of the following possibilities occurs:

(i) The element a is the maximum of the set A, and we have $\bar{A}_a = A$.

(ii) There exists an element b of the set A such that

$$\bar{A}_a = A_b = \{x \in A \mid x < b\}.$$

Proof. Obviously, the set $\bar{A}_a$ is an initial part of the set A. The assertion follows from Theorem 7.1. □

7.3 Corollary. *Let A be a well-ordered set, and let a be an element of the set A. Then the element a is the maximum of the set A, or it has a successor b.*

Proof. The assertion follows from Theorem 7.2. □

7.4 Theorem. *Let A be a well-ordered set, and let $(A_i)_{i\in I}$ be a family of initial segments of the set A. Then the set $\bigcup_{i\in I} A_i$ is either an initial segment of the set A, or we have*

$$A = \bigcup_{i\in I} A_i.$$

Proof. Let

$$B := \bigcup_{i\in I} A_i.$$

Obviously, the set B is an initial part of the set A. The assertion follows from Theorem 7.1. □

7.5 Theorem. *Let A be a well-ordered set, and let*

$$B := \bigcup_{x\in A} A_x.$$

The one of the following possibilities occurs:

(i) The set A has a maximum m, and we have $B = A_m$.

(ii) The set A has no maximum, and we have $B = A$.

Proof. By Theorem 7.4, we have $B = A$ or $B = A_a$ for an element a of the set A. We have to show that in the second case the element a is the maximum of the set A. Otherwise, there exists an element z of the set A such that $a < z$. It follows that

$$a \in A_z \subseteq \bigcup_{x\in A} A_x = B = A_a = \{x \in A \mid x < a\},$$

a contradiction. □

7.6 Examples. (a) If $A = (\mathbb{N}_0, \leqslant)$ with the standard order, then we have

$$B := \bigcup_{x\in A} A_x = \bigcup_{n\in\mathbb{N}_0} A_n = \mathbb{N}_0 = A.$$

(b) If $A = \{1, 3, 5\}$, then we have

$$B := \bigcup_{x\in A} A_x = A_5.$$

7.7 Corollary. *Let $(A, \leqslant)$ be a well-ordered set, and let b be an element of the set A. Then exactly one of the following possibilities occurs:*

(i) The element b has no predecessor. In this case we have

$$A_b = \bigcup_{z<b} A_z.$$

(ii) The element b has a (unique) predecessor a. In this case the predecessor a is the maximal element of the initial segment $A_b := \{x \in A \mid x < b\}$, and we have

$$A_b = \bar{A}_a := \{x \in A \mid x \leqslant a\} \text{ and } A_b = A_a \cup \{a\}.$$

Proof. Since the pair $(A, \leqslant)$ is well-ordered, the subset A_b of the set A is also well-ordered (Proposition 5.6). Let

$$B := \bigcup_{z<b} A_z = \bigcup_{z \in A_b} (A_b)_z.$$

By Theorem 7.5, we have $B = A_b$, or the set A_b has a maximal element a. In the second case we have

$$A_b = \bar{A}_a := \{x \in A \mid x \leqslant a\} \text{ and } A_b = A_a \cup \{a\}.$$

In particular the element b is a successor of the element a. □

7.8 Definition. Let $(A, \leqslant)$ be a well-ordered set, and let a be an element of the set A which has no predecessor. Then the number a is called a **limit number**.

French / German. Limit number = Ordinale limite = Limeszahl or Grenzzahl.

Historical Notes:

Initial and final segments have been introduced by Cantor:

Ist f irgend ein vom Anfangselement f_1 verschiedenes Element der wohlgeordneten Menge F, so wollen wir die Menge A aller Elemente von F, welche $< f$, einen **Abschnit von** F *und zwar den* **durch das Element** f **bestimmten Abschnitt von** F *nennen. Dagegen heiße die Menge R aller übrigen Elemente von F mit Einschluss von f ein* **Rest von** F *und zwar der* **durch das Element** f **bestimmte Rest von** F*.*

See [Cantor 1897, p. 210].

If f is an element of the well-ordered set F which is different from the minimal element of the set F, then we call the set of all elements of the set F which are $< f$ a **segment of** F*, more precisely the* **segment of** F **defined by** f*. The set R of the other elements including f is called the* **rest of** F*, more precisely the* **rest of** F **defined by the element** f*.*

Note that the empty set is not considered as an initial segment by Cantor. At that time the empty set often was not considered as a subset and had to be treated separately.

Corollary 7.7 is due to Cantor:

K. *Jede Zahl α der zweiten Zahlenklasse ist entweder derart, dass sie aus einer nächstkleineren α_1 durch Hinzufügung der 1 hervorgeht*

$$\alpha = \alpha_1 + 1,$$

oder es lässt sich eine Fundamentalreihe $(\alpha_\nu)_\nu$ von Zahlen der ersten oder der zweiten Zahlenklasse angeben, so dass

$$\alpha = \lim_\nu a_\nu.$$

See [Cantor 1897, p. 225].

K. *Every number α of the second counting class is either of the form that it results from the previous α_1 by adding 1,*

$$\alpha = \alpha_1 + 1,$$

or it exists a fundamental series $(\alpha_\nu)_\nu$ *of numbers of the first or the second counting class such that*

$$\alpha = \lim_\nu a_\nu.$$

The numbers of the first counting class are the finite numbers, the numbers of the second counting class are the infinite numbers.

Theorem K of Cantor deals with ordinal numbers which are specific well-ordered sets (see Unit *Ordinal Numbers* [Garden 2020g] for more details). For ordinal numbers the successor of an element α_1 is the element $\alpha_1^+ := \alpha_1 \cup \{\alpha_1\}$ abbreviated by Cantor by $\alpha_1 + 1$.

The fundamental series $(\alpha_\nu)_\nu$ consists of the initial segments $(A_\nu)_{\nu<\alpha}$, and the limit is defined by

$$\lim_\nu a_\nu = \lim_\nu A_\nu := \bigcup_{\nu<\alpha} A_\nu.$$

8 The Principle of Transfinite Induction

The Principle of Transfinite Induction:

8.1 Theorem. (Principle of Transfinite Induction) *Let* $(A, \leqslant)$ *be a well-ordered set. For an element* z *of the set* A*, we denote by*

$$A_z := \{x \in A \mid x < z\}$$

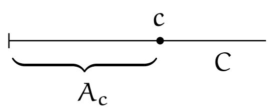
the initial segment of the set A *with respect to the element* z*. Let* B *be a subset of the set* A *fulfilling the following condition:*

$A_z \mapsto z$*: For all elements* z *of the set* A*, it follows from* $A_z \subseteq B$ *that the element* z *is contained in the set* B*.*

Then we have $B = A$*.*

Proof. Suppose that the set B is a subset of the set A fulfilling the condition $A_z \mapsto z$ for all elements z of the set A, and assume that $B \neq A$.

Since $B \neq A$, the set $C := A \setminus B$ is non-empty. Since the set A is well-ordered, there exists a minimal element c of the set C.



Let $A_c := \{a \in A \mid a < c\}$ be the initial segment of the set A with respect to the element c. Since the element c is the minimal element of the set C, it follows that $A_c \cap C = \emptyset$, that is, the set A_c is a subset of the set B. It follows from the condition $A_z \mapsto z$ (with $z := c$) that the element c is contained in the set B, in contradiction to

$$c \in C = A \setminus B.$$

Hence, we have $B = A$. □

French / German. Transfinite Induction = Récurrence transfinie = Transfinite Induktion.

8.2 Remark. Let $(A, \leqslant)$ be a well-ordered set, and let m be the minimal element of the set A. For $z := m$, the condition $A_z \mapsto z$ of the principle of transfinite induction (Theorem 8.1) becomes the condition $\emptyset \mapsto m$.

The condition $\emptyset \mapsto m$ simply means that the minimal element m is contained in the set B.

The Second Principle of Induction:

The principle of transfinite induction applied on the set $\mathbb{N}_0$ yields a variation of the principle of induction:

8.3 Theorem. (Second Principle of Induction) *Suppose that $(A_n)_{n \in \mathbb{N}_0}$ is a family of sentences with the following properties:*

(i) $n = 0$: The sentence A_0 is true.

(ii) $\{0, 1, \ldots, n\} \mapsto n+1$: If the sentence A_j is true for all numbers $j = 0, 1, \ldots, n$, then the sentence A_{n+1} is also true.

Then every sentence of the family $(A_n)_{n \in \mathbb{N}_0}$ is true.

Proof. Let $(A_n)_{n \in \mathbb{N}_0}$ be a family of sentences fulfilling Conditions (i) and (ii). Let

$$B := \{n \in \mathbb{N}_0 \mid A_n \text{ is true}\}.$$

We have to show that $B = \mathbb{N}_0$: For, let n be a natural number, and let

$$S_n := \{m \in \mathbb{N}_0 \mid m < n\}$$

be the initial segment of the set $\mathbb{N}_0$ with respect to the element n.

Note that $S_0 = \emptyset$. The Condition $S_0 \mapsto 0$ just means that the number 0 is an element of the set B. This assertion follows from Condition (i).

Condition (ii) implies Condition $S_{n+1} \mapsto n+1$. Hence, the Condition $S_n \mapsto n$ is fulfilled for all natural numbers.

Since the set $\mathbb{N}_0$ is well-ordered (Theorem 5.4), it follows from Theorem 8.1 that $B = \mathbb{N}_0$. □

French / German. Second Principle of Induction = Raisonnement par Récurrence = Prinzip der vollständigen Induktion. (It seems that there is no specific notion for the *second* principle of induction in French or in German.)

Historical Notes:

The Principle of Transfinite Induction has been used by Cantor for the first time:

[...] Denn gäbe es Ausnahmewerte von ξ, [...], so müsste nach Satz B, §16 einer derselben, wir nennen ihn α, der kleinste sein. Es wäre dann der Satz gültig für $\xi < \alpha$, nicht aber für $\xi \leqslant \alpha$, was mit dem soeben Bewiesenen in Widerspruch stehen würde. [...]

See [Cantor 1897, p. 232].

[...] Assume that there exist exceptional values of ξ *[...]. Then, by Theorem B, §16, one of these, say* α*, should be the smallest. The theorem would then be valid for* $\xi < \alpha$*, but not for* $\xi \leqslant \alpha$*, which would be in contradiction with what has just been proven. [...]*

This is a small piece of a proof of a theorem of Cantor illustrating that he was aware of the principle of transfinite induction. It was Felix Hausdorff who gave the principle of transfinite induction its explicit form:

Eine Aussage $f(\alpha)$ *bezüglich der Ordnungszahl* α *ist für jedes* α *richtig, sobald* $f(0)$ *richtig ist und sobald aus der Richtigkeit aller* $f(\xi)$ *für* $\xi < \alpha$ *auf die Richtigkeit von* $f(\alpha)$ *geschlossen werden kann.*

See [Hausdorff 1914, p. 113].

A statement $f(\alpha)$ *regarding the ordinal number* α *is correct for every* α *as soon as* $f(0)$ *is correct and as soon as it can be concluded from* $f(\xi)$ *for all* $\xi < \alpha$ *that* $f(\alpha)$ *is correct.*

Note that Hausdorff formulates the principle of transfinite induction only for ordinal numbers. Ordinal numbers are specific well-ordered sets which will be explained in Unit *Ordinal Numbers* [Garden 2020g]. However, since every well-ordered set is isomorphic to an ordinal number, this restriction is not important.

9 The Well-Ordering Theorem

The main result of this unit is Theorem 9.7 saying that every set can be well-ordered. We will present two different proofs for this result. One proof is based on the Lemma of Zorn, the other proof is based on the axiom of choice. For the first proof we need some results about ordered sets which are explained in detail in Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. These results are recalled in Proposition 9.1 to Proposition 9.5. In addition, we recall the Lemma of Zorn (see Theorem 9.6).

Chains of Ordered Sets:

9.1 Proposition. *Let* $(A, \leqslant_A)$*,* $(B, \leqslant_B)$ *and* $(C, \leqslant_C)$ *be three ordered sets fulfilling the following conditions:*

(i) The set A *is a subset of the set* B*, and the set* B *is a subset of the set* C*.*

(ii) The order $\leqslant_A$ *is induced by the order* $\leqslant_B$*, that is, we have*

$$x \leqslant_A y \textit{ if and only if } x \leqslant_B y \textit{ for all } x, y \in A.$$

(iii) The order $\leqslant_B$ *is induced by the order* $\leqslant_C$*, that is, we have*

$$x \leqslant_B y \textit{ if and only if } x \leqslant_C y \textit{ for all } x, y \in B.$$

(a) The order $\leqslant_A$ *is induced by the order* $\leqslant_C$*, that is, we have*

$$x \leqslant_A y \textit{ if and only if } x \leqslant_C y \textit{ for all } x, y \in A.$$

(b) If there exist two elements b *and* c *of the sets* B *and* C*, respectively, such that*

$$\begin{aligned} A &= \{x \in B \mid x <_B b\} \text{ and} \\ B &= \{x \in C \mid x <_C c\}, \end{aligned}$$

then we have $A = \{x \in C \mid x <_C b\}$.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

9.2 Definition. Let I be a totally ordered index set. A family $(A_i, \leqslant_i)_{i \in I}$ is called a **chain of ordered sets** if the following conditions are fulfilled:

(i) The family $(A_i)_{i \in I}$ is a chain, that is, we have $A_i \subseteq A_j$ whenever $i \leqslant j$.

(ii) For each two elements i and j of the set I such that $i \leqslant j$, the order $\leqslant_i$ on the set A_i is induced by the order $\leqslant_j$ on the set A_j, that is, we have

$$x \leqslant_i y \text{ if and only if } x \leqslant_j y \text{ for all } x, y \in A_i.$$

9.3 Proposition. *Let* I *be a totally ordered index set, let* $(A_i, \leqslant_i)_{i \in I}$ *be a* **chain of ordered sets***, and let* $A := \bigcup_{i \in I} A_i$.

(a) There exists exactly one order $\leqslant$ *on the set* A *such that for all elements* i *of the set* I*, we have*

$$x \leqslant y \text{ if and only if } x \leqslant_i y \text{ for all } x, y \in A_i,$$

that is, the orders $\leqslant_i$ *are induced by the order* $\leqslant$ *for all elements* i *of the set* I.

(b) If we have $A = A_i$ *for an element* i *of the set* I*, then we have* $(A, \leqslant) = (A, \leqslant_i)$.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

9.4 Definition. Let I be a totally ordered index set, let $(A_i, \leqslant_i)_{i \in I}$ be a chain of ordered sets, and let $A := \bigcup_{i \in I} A_i$.

The order $\leqslant$ on the set A defined in Proposition 9.3 is called the **order of the set** $A = \bigcup_{i \in I} A_i$ **induced by the chain of ordered sets** $(A_i, \leqslant_i)_{i \in I}$.

9.5 Proposition. *Let* I *be a totally ordered index set, and let* $(A_i, \leqslant_i)_{i \in I}$ *be a chain of ordered sets fulfilling the following condition:*

If i *and* j *are two elements of the set* I *such that* $i < j$*, then we have* $(A_i, \leqslant_i) = (A_j, \leqslant_j)$*, or there exists an element* b_{ij} *of the set* A_j *such that* $A_i = \{x \in A_j \mid x <_j b_{ij}\}$.

Let $A := \bigcup_{i \in I} A_i$*, and let* $\leqslant$ *be the order on the set* A *induced by the chain* $(A_i, \leqslant_i)_{i \in I}$ *of ordered sets. Then for each element* i *of the set* I*, one of the following possibilities occurs:*

(i) We have $(A_i, \leqslant_i) = (A, \leqslant)$.

(ii) There exists an element b_i *of the set* A *such that* $A_i = \{x \in A \mid x < b_i\}$.

In other words, for each element i *of the set* I*, the set* A_i *is either an initial segment of the set* A*, or we have* $A_i = A$.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

The Lemma of Zorn:

Note that a **chain** in a (partially) ordered set A is a totally ordered subset of the set A.

9.6 Theorem. (Lemma of Zorn) *Let $A = (A, \leqslant)$ be a (partially) ordered set such that every chain C of the set A has an upper bound in the set A.*
Then the set A contains a maximal element.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

The Well-Ordering Theorem:

9.7 Theorem. (Zermelo) *Every set A can be well-ordered, that is, there exists an order $\leqslant_A$ on the set A such that the pair $(A, \leqslant_A)$ is well-ordered.*

French / German. Well-ordering theorem = Théorème de Zermelo = Wohlordnungssatz.

9.8 Remark. One proof idea for the proof of the well-ordering theorem could be as follows: Let

$$\mathcal{A} := \big\{(X, \leqslant_X) \mid X \subseteq A \text{ and } (X, \leqslant_X) \text{ is well-ordered}\big\}$$

be the set of the pairs $(X, \leqslant_X)$ where X is a subset of the set A such that there exists a well-ordering $\leqslant_X$ on the set X. Define an order $\leqslant$ on the set $\mathcal{A}$ by

$$(X, \leqslant_X) \leqslant (Y, \leqslant_Y) \text{ if and only if } X \subseteq Y \text{ and } \leqslant_X \text{ is induced by } \leqslant_Y .$$

Show, by using the Lemma of Zorn, that the set $\mathcal{A}$ admits a maximal element $(M, \leqslant_M)$ and show that $M = A$. As a consequence the pair $(A, \leqslant_A) = (M, \leqslant_M)$ is well-ordered.

However, this idea does not work: For, let $A := \mathbb{N}$, let

$$B_n := \{1, 2, \ldots, n\} \text{ for all } n \in \mathbb{N},$$

and define the order $\leqslant_n$ on the set B_n by

$$n \leqslant_n \ldots \leqslant_n 2 \leqslant_n 1.$$

By Theorem 5.3, the pairs $(B_n, \leqslant_n)$ are well-ordered for each natural number n. In order to apply the Lemma of Zorn, we have to look at the set

$$B := \bigcup_{n \in \mathbb{N}} B_n$$

and at the order $\leqslant_B$ such that each order $\leqslant_n$ on the set B_n is induced by the order $\leqslant_B$. It follows that $B = \mathbb{N}$ and that

$$\ldots \leqslant_B n + 1 \leqslant_B n \leqslant_B \ldots \leqslant_B 2 \leqslant_B 1.$$

To apply the Lemma of Zorn the pair $(B, \leqslant_B)$ should be an element of the set $\mathcal{A}$, that is, it should be well-ordered. But this is not the case since the pair $(B, \leqslant_B)$ does not have a minimal element. This example explains why we will make use of the initial segments in the proof of Theorem 9.7.

Proof. We will apply the Lemma of Zorn (Theorem 9.6).

The proof is organized as follows: In Step 1 we consider the set

$$W := \{(X, \leqslant) \mid X \subseteq A \text{ and } (X, \leqslant) \text{ is well-ordered}\},$$

and we define a partial order $\leqslant_W$ on the set W (Step 2 and 3). We then show that the set W is non-empty (Step 4) and that every chain in the set $W = (W, \leqslant_W)$ has an upper bound in the set W (Step 5). By the Lemma of Zorn, the set $W = (W, \leqslant_W)$ has a maximal element M (Step 6). Finally, we will see in Step 7 that $M = A$ implying that the set A is well-ordered.

Step 1. *Definition of the set W:*

Let

$$W := \{(X, \leqslant) \mid X \subseteq A \text{ and } (X, \leqslant) \text{ is well-ordered}\}.$$

Note that the elements of the set W are the pairs $(X, \leqslant)$ where the set X is a subset of the set A and where $\leqslant$ is an order on the set X such that the pair $(X, \leqslant)$ is well-ordered. If the set A contains two elements x and y, then the pairs $(\{x, y\}, \leqslant_1)$ and $(\{x, y\}, \leqslant_2)$ with $x \leqslant_1 y$ and $y \leqslant_2 x$ are both contained in the set W.

This example shows that a set X can appear several times in the set W, of course with different orders.

Step 2. *Definition of a partial order $\leqslant_W$ on the set W:*

Let $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ be two elements of the set W. We set

$$(X, \leqslant_X) \leqslant_W (Y, \leqslant_Y)$$

if the following conditions are fulfilled:

(i) We have $X \subseteq Y$.

(ii) For each two elements x and y of the set X, we have

$$x \leqslant_X y \text{ if and only if } x \leqslant_Y y,$$

that is, the order $\leqslant_X$ of the set X is induced by the order $\leqslant_Y$ of the set Y (see Definition 2.2).

(iii) We have $(X, \leqslant_X) = (Y, \leqslant_Y)$, or the set X is an initial segment of the set Y, that is, there exists an element b of the set Y such that

$$X = \{x \in Y \mid x <_Y b\} \text{ (see Definition 2.6).}$$

Note that Property (iii) will be used in Step 5.

Step 3. *The relation $\leqslant_W$ on the set W defined in Step 2 is an order:*

Step 3.1. *The relation $\leqslant_W$ is reflexive:*

The proof is obvious.

Step 3.2 *The relation $\leqslant_W$ is antisymmetric:*

Suppose that $(X, \leqslant_X) \leqslant_W (Y, \leqslant_Y)$ and $(Y, \leqslant_Y) \leqslant_W (X, \leqslant_X)$ for two elements $(X, \leqslant_X)$ and $(Y, \leqslant_Y)$ of the set W. It follows from Condition (i) of Step 2 that the set X is a subset of the set Y and that the set Y is a subset of the set X, that is, $X = Y$.

It follows from Condition (ii) of Step 2 that

$$x \leqslant_X y \text{ if and only if } x \leqslant_Y y \text{ for all } x, y \in X = Y.$$

It follows that $(X, \leqslant_X) = (Y, \leqslant_Y)$.

Step 3.3 *The relation $\leqslant_W$ is transitive:*

Let $(X, \leqslant_X)$, $(Y \leqslant_Y)$ and $(Z, \leqslant_Z)$ be three elements of the set W such that

$$(X, \leqslant_X) \leqslant_W (Y, \leqslant_Y) \text{ and } (Y, \leqslant_Y) \leqslant_W (Z, \leqslant_Z).$$

Then the ordered sets $(X, \leqslant_X)$, $(Y, \leqslant_Y)$ and $(Z, \leqslant_Z)$ fulfill the assumptions of Proposition 9.1, and it follows from Proposition 9.1 that $(X, \leqslant_X) \leqslant_W (Z, \leqslant_Z)$. Hence, the relation $\leqslant_W$ is transitive.

Step 4. *We have $W \neq \emptyset$:*

By Proposition 5.2, the pair $(\emptyset, \emptyset)$ is well-ordered and therefore an element of the set W.

Step 5. *Let $\mathcal{C} := \{(C_i, \leqslant_i) \mid i \in I\}$ be a chain in the set W for some index set I. Let*

$$C := \bigcup_{i \in I} C_i,$$

and let $\leqslant_C$ be the order on the set C induced by the chain $\{(C_i, \leqslant_i) \mid i \in I\}$ (Definition 9.4). Then the pair $(C, \leqslant_C)$ is contained in the set W, and we have

$$(C_i, \leqslant_i) \leqslant_W (C, \leqslant_C) \textit{ for all } i \in I:$$

Step 5.1. *The set C is a subset of the set A:*

Since each set C_i is a subset of the set A, the set $C = \bigcup_{i \in I} C_i$ is also a subset of the set A.

Step 5.2. *We have*

$$(C_i, \leqslant_i) \leqslant_W (C, \leqslant_C) \textit{ for all } i \in I.$$

In particular, for each element i of the set I, there exists an element c_i of the set C such that

$$C_i = \{x \in C \mid x <_C c_i\}:$$

The assertion follows from Proposition 9.5.

Step 5.3. *The pair $(C, \leqslant_C)$ is well-ordered:*

We will apply Theorem 6.1: Let B be an initial part of the set C with $B \neq C$. Since $B \neq C$, there exists an element c of the set C such that

$$x < c \text{ for all } x \in B.$$

Since $C = \bigcup_{i \in I} C_i$, there exists an element i of the set I such that the element c is contained in the set C_i. It follows that the set B is a subset of the set C_i. Since the set B is an initial part of the set C, it is also an initial part of the set C_i.

Since the pair $(C_i, \leqslant)$ is well-ordered, it follows from Theorem 6.1 that the initial part B is an initial segment of the set C_i. It follows from Proposition 9.5 that the set B is also an initial segment of the set C.

By Theorem 6.1, the set pair $(C, \leqslant)$ is well-ordered.

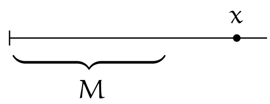
Step 6. *The pair $(W, \leqslant_W)$ admits a maximal element:*

By Step 5, every chain of the set W admits an upper bound. It follows from the Lemma of Zorn (Theorem 9.6) that the pair $(W, \leqslant_W)$ admits a maximal element.

Step 7. *Let* $(M, \leqslant_M)$ *be a maximal element of the set* W. *Then we have* $M = A$. *In particular the pair* $(M, \leqslant_M) = (A, \leqslant_M)$ *is well-ordered:*

Assume that $M \neq A$. Then there exists an element x of the set A which is not contained in the set M.

Let $M_x := (M_x, \leqslant_{M_x})$ with $M_x := M \cup \{x\}$ be the one point extension of the set M as defined in Definition 2.7.



By Theorem 5.8, the pair $(M \cup \{x\}, \leqslant_{M_x})$ is well-ordered. Hence, the pair $(M_x, \leqslant_{M_x})$ is contained in the set W.

We claim that $(M, \leqslant_M) <_W (M_x, \leqslant_{M_x})$:

Condition (i) of Step 2: By definition, we have

$$M \subset M \cup \{x\} = M_x.$$

Condition (ii) of Step 2: The assertion follows directly from the definition of the order $\leqslant_{M_x}$ on the set M_x.

Condition (iii) of Step 2: Since $M_x = M \cup \{x\}$ and since

$$z <_{M_x} x \text{ for all } z \in M,$$

we have $M = \{z \in M_x \mid z <_{M_x} x\}$.

Finally, we have $M \neq M_x$ since the element x of the set A is contained in the set M_x, but not in the set M.

The assertion $(M, \leqslant_M) <_W (M_x, \leqslant_{M_x})$ is a contradiction to the fact that the pair $(M, \leqslant_M)$ is a maximal element of the set W. $\square$

Historical Notes:

Cantor was the first to formulate the well-ordering theorem:

Der Begriff der **wohlgeordneten Menge** *weist sich als fundamental für die ganze Mannigfaltigkeitslehre aus. Dass es immer möglich ist, jede wohldefinierte Menge in die Form einer wohlgeordneten Menge zu bringen, auf dieses, wie mir scheint, grundlegende und folgenreiche, durch seine Allgemeingültigkeit besonders merkwürdige Denkgesetz werde ich in einer späteren Abhandlung zurückkommen.*

See [Cantor 1883, p. 550].

The concept of the well-ordered set proves to be fundamental for the whole theory of sets. That it is always possible to bring every well-defined set into the form of a well-ordered set, I shall come back to this, it seems to me, fundamental and consequential law of thought, which is particularly remarkable because of its general validity.

Cantor expressed his conviction that every set can be well-ordered and even that it is a fundamental law of mathematics. However, he did never prove it.

The Well-Ordering Theorem (Theorem 9.7) has been proven by Ernst Zermelo:

Beweis, dass jede Menge wohlgeordnet werden kann.

See [Zermelo 1904, p. 514].

Proof that every set can be well-ordered.

This is the title of the article [Zermelo 1904] with Zermelo's first proof of the well-ordering theorem. He published a second proof in 1908 [Zermelo 1908] which is explained in Section 11. The proof of the well-ordering theorem explained above (Theorem 9.7) is based on the Lemma of Zorn published by Max Zorn in 1935 [Zorn 1935].

In 1900, David Hilbert [Hilbert 1900] gave a talk at the International Congress of Mathematics in Paris and presented a selection of 23 problems that he considered to be particularly important. These problems became famous as Hilbert's 23 problems. The first problem is closely related to Ernst Zermelo's well-ordering theorem:

The first of Hilbert's problems is to prove or disprove Canotor's continuum hypothesis. This hypothesis states that every infinite subset of the real numbers has the cardinality of the set $\mathbb{N}_0$ of the natural numbers or the cardinality of the set $\mathbb{R}$ of the real numbers. Interestingly enough, this problem was solved by Paul Cohen in 1963/64 [Cohen 1963], [Cohen 1964] in that the axioms of Zermelo and Fraenkel are not strong enough to decide this question. The axioms of Zermelo and Fraenkel can be supplemented with the continuum hypothesis, but also with its opposite as an additional axiom.

In this context Hilbert mentions the well-ordering theorem as an open question, and he finds in desirable that the problem be resolved:

Es erhebt sich nun die Frage, ob sich die Gesamtheit aller Zahlen nicht in anderer Weise so ordnen lässt, dass jede Teilmenge ein frühestes Element hat, d.h. ob das Kontinuum auch als wohlgeordnete Menge aufgefasst werden kann, was Cantor bejahen zu müssen glaubt. Es erscheint mir höchst wünschenswert, einen direkten Beweis dieser merkwürdigen Behauptung von Cantor zu gewinnen, etwa durch wirkliche Angabe einer solchen Ordnung der Zahlen, bei welcher in jedem Teilsystem eine früheste Zahl aufgewiesen werden kann.

See [Hilbert 1900, p. 264].

The question now arises whether the totality of all numbers cannot be ordered in some other way in such a way that every subset has a minimal element, i.e. whether the continuum can also be understood as a well-ordered set, which Cantor believes he must affirm. It seems to me highly desirable to obtain a direct proof of this remarkable assertion by Cantor, for example by actually specifying such an order of numbers in which a minimal number can be shown in each subsystem.

On the one hand, Zermelo proved even more. He showed the well-ordering theorem not only for the set of the real umbers, but for every set. On the other hand, the proof of Zermelo (and all other known proofs of the well-ordering theorem) is a pure existence proof. No such order has been explicitly specified for an arbitrary set or for the set of the real numbers.

10 Final Segments in Well-Ordered Sets

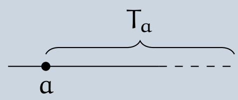
The second proof of the well-ordering theorem will be based on final segments. To motivate the proof idea we will investigate final segments in well-ordered sets.

Analysis of the Final Segments T_a:

10.1 Proposition. *Let A be a well-ordered set. For each element a of the set A let*

$$T_a := \{x \in A \mid x \geqslant a\}$$

be the final segment of the set A with respect to the element a.



Given two elements x and y of the set A we have

$$x \leqslant y \text{ if and only if } T_y \subseteq T_x. \tag{1}$$

Proof. The proof is obvious. □

10.2 Remark. The second proof of the well-ordering theorem is based on the following observation:

The idea of the proof is first to construct the set

$$\mathcal{T} := \{T_x \mid x \in A\} \cup \{\emptyset\}$$

and then to define an order $\leqslant$ on the set A using Condition (1) of Proposition 10.1. It then will turn out that this order is a well-ordering on the set A.

We first analyze the set $\mathcal{T}$ as a subset of a well-ordered set A. If A is a well-ordered set, there exists the following choice function

$$\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A, \ \alpha : X \mapsto \min(X)$$

where $\min(X)$ denotes the minimal element of the set X.

Note that we have

$$\alpha(T_a) := a \text{ for all } a \in A.$$

It will turn out in Proposition 10.6 that the set $\mathcal{T}$ is a minimal θ-chain (Definition 10.3) with respect to the pair (A, α).

10.3 Definition. Let A be non-empty set, and let $\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$ be a choice function on the set A.

The set $\mathcal{A}$ is called a **θ-chain with respect to the pair** (A, α) if it fulfills the following conditions:

(i) The set A and the empty set $\emptyset$ are elements of the set $\mathcal{A}$.

(ii) If the set $X \neq \emptyset$ is an element of the set $\mathcal{A}$, then the set $X' := X \setminus \{\alpha(X)\}$ is also an element of the set $\mathcal{A}$.

(iii) If $(X_i)_{i \in I}$ is a family of sets of the set $\mathcal{A}$ for some index set $I \neq \emptyset$, then we have

$$\bigcap_{i \in I} X_i \in \mathcal{A}.$$

French / German. Chain = Chaîne = Kette.

For the proof of Proposition 10.6 we will need the following results:

10.4 Proposition. *Let A be a totally ordered set, let a be an element of the set A, and let*

$$A_a := \{x \in A \mid x < a\} \text{ (initial segment) and } T_a := \{x \in A \mid x \geqslant a\}.$$

(a) We have $A = A_a \mathbin{\dot\cup} T_a$.

(b) We have $(A_a)^c = T_a$ (complement of the set A_a in the set A).

(c) We have $(T_a)^c = A_a$ (complement of the set T_a in the set A).

Proof. The proof is obvious. □

10.5 Proposition. *Let $(X_i)_{i \in I}$ be a family of sets. Then we have*

$$\Big(\bigcup_{i \in I} X_I\Big)^c = \bigcap_{i \in I} (X_i)^c.$$

Proof. For the proof see Unit *Families and the Axiom of Choice Sets* [Garden 2020b]. □

10.6 Proposition. *Let $A \neq \emptyset$ be a well-ordered set. For each element a of the set A, let*

$$T_a := \{x \in A \mid x \geqslant a\}$$

be the final segment of the set A with respect to the element a. Let

$$\mathcal{T} := \{T_a \mid a \in A\} \cup \{\emptyset\},$$

and let

$$\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A, \ \alpha : X \mapsto \min(X)$$

where $\min(X)$ denotes the minimal element of the set X.

(a) We have

$$\alpha(T_a) := a \text{ for all } a \in A.$$

(b) The set $\mathcal{T}$ is a θ-chain with respect to the pair (A, α).

(c) Let $\mathcal{S}$ be a further θ-chain with respect to the pair (A, α) such that $\mathcal{S} \subseteq \mathcal{T}$. Then we have $\mathcal{S} = \mathcal{T}$.

Proof. (a) is obvious.

(b) We have to verify Conditions (i) to (iii) of Definition 10.3:

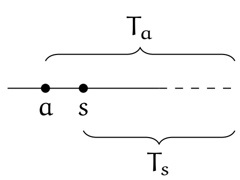
(i) Let m be the minimal element of the set A. Then we have

$$A = T_m = \{x \in A \mid x \geqslant m\} \in \mathcal{T}.$$

By definition, the empty set is contained in the set $\mathcal{T}$.

(ii) Let $T := T_a$ be an element of the set $\mathcal{T}$. Let $S := T \setminus \{a\}$. Since the set A is well-ordered, the set S has a minimal element s. It follows that

$$S = T_s \in \mathcal{T}.$$



(iii) Let $(T_i)_{i \in I}$ be a family of sets of the set $\mathcal{T}$ over an index set $I \neq \emptyset$. For each element i of the set I, let $A_i := (T_i)^c$ be the complement of the set T_i in the set A. It follows from Proposition 10.4 that each set A_i is an initial segment of the set A.

By Theorem 7.4, the set

$$U := \bigcup_{i \in I} A_i$$

is either an initial segment of the set A or we have $U = A$.

If the set U is an initial segment of the set A, then it follows from Proposition 10.4 that the complement U^c of the set U in the set A is an element of the set $\mathcal{T}$.

If $U = A$, then we have

$$U^c = \emptyset \in \mathcal{T}.$$

It follows from Proposition 10.5 that

$$\bigcap_{i \in I} T_i = \Big(\big(\bigcap_{i \in I} T_i \big)^c \Big)^c = \Big(\bigcup_{i \in I} (T_i)^c \Big)^c = \Big(\bigcup_{i \in I} A_i \Big)^c = U^c \in \mathcal{T}.$$

(c) Let $\mathcal{S}$ be a θ-chain and suppose that the θ-chain $\mathcal{S}$ is a subset of the θ-chain $\mathcal{T}$. Assume that $\mathcal{S} \neq \mathcal{T}$.

Then there exists a set T of the θ-chain $\mathcal{T}$ not contained in the θ-chain $\mathcal{S}$.

Step 1. *We have* $T \neq \emptyset$*:*

By definition, the empty set is contained in a θ-chain, hence the empty set is contained in the set $\mathcal{S}$.

Step 2. *Definition of the set* Z*:*

Let

$$Z := \{x \in A \mid T_x \notin \mathcal{S}\}.$$

By assumption, we have $Z \neq \emptyset$. Since the set A is well-ordered, there exists a minimal element z of the set Z. By Corollary 7.7, we may distinguish the following cases:

Case 1. *The element* z *has no predecessor in the set* A*, and the element* z *is the minimal element of the set* A*:*

By Condition (i) of Definition 10.3, we have

$$T_z = \{x \in A \mid x \geqslant z\} = A \in \mathcal{S},$$

in contradiction to the assumption that the element z is contained in the set Z.

Case 2. *The element* z *has no predecessor in the set* A*, and the element* z *is not the minimal element of the set* A*:*

It follows from Corollary 7.7 that

$$A_z = \bigcup_{x<z} A_x.$$

Since $z \neq \min A$, the set $\{x \in A \mid x < z\}$ is non-empty. Since the element z is the minimal element of the set $\mathcal{S}$, each set T_x is an element of the set $\mathcal{T}$ for $x < z$. By Condition (iii) of Definition 10.3 and Proposition 10.5, we obtain

$$T_z = (A_z)^c = \Big(\bigcup_{x<z} A_x \Big)^c = \bigcap_{x<z} (A_x)^c = \bigcap_{x<z} T_x \in \mathcal{S},$$

a contradiction. (Note that the intersection $\bigcap_{x<z} T_x$ is only defined if the set $\{x \in A \mid x < z\}$ is non-empty.)

Case 3. *The element z has a predecessor y in the set A.*

Since z is the minimal element of the set Z, if follows from $y < z$ that the set T_y is contained in the set $\mathcal{S}$. Since the set $\mathcal{S}$ is a θ-chain, it follows from Condition (ii) of Definition 10.3 that

$$T_z = T_y \setminus \{y\} \in \mathcal{S},$$

in contradiction to the fact that the element z belongs to the set Z. □

Historical Notes:

The results of this section are at least implicitly contained in [Zermelo 1908]. See also the historical notes at the end of Section 11.

11 A Second Proof of the Well-Ordering Theorem

Existence of Minimal θ-Chains:

We recall that given a non-empty set A the axiom of choice guarantees the existence of a so-called **choice function**, that is, of a function

$$\alpha : \mathcal{P}(A) \to A$$

such that

$$\alpha(X) \in A \text{ for all } X \in \mathcal{P}(A) \setminus \{\emptyset\}.$$

For the second proof of the well-ordering theorem we will need the following result:

11.1 Proposition. *Let A be a non-empty set, and let*

$$\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A$$

be a choice function on the set A.

Then there exists a minimal θ-chain $\mathcal{T}$ with respect to the pair (A, α).

Proof. Step 1. *There exists at least one θ-chain:*

Obviously, the power set $\mathcal{P}(A)$ of the set A is a θ-chain.

Step 2. *Definition of the θ-chain $\mathcal{T}$:*

Let

$$\mathcal{T} := \bigcap \{\mathcal{A} \subseteq \mathcal{P}(A) \mid \mathcal{A} \text{ is a } \theta\text{-chain}\}.$$

One easily verifies that the set $\mathcal{T}$ is a θ-chain.

Step 3. *The θ-chain $\mathcal{T}$ is minimal, that is, if a θ-chain $\mathcal{Z}$ is a subset of the set $\mathcal{T}$, then we have $\mathcal{Z} = \mathcal{T}$:*

The assertion follows from

$$\mathcal{Z} \subseteq \mathcal{T} = \bigcap \{\mathcal{A} \mid \mathcal{A} \text{ is a } \theta\text{-chain}\} \subseteq \mathcal{Z}.$$

□

We will need the following proposition:

11.2 Proposition. *Let* A *be a non-empty set, let*

$$\alpha : \mathcal{P}(A) \to A$$

be a choice function, that is, a mapping from the power set $\mathcal{P}(A)$ *of the set* A *into the set* A *such that*

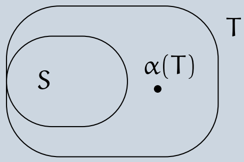
$$\alpha(X) \in X \text{ for all } X \in \mathcal{P}(A) \setminus \{\emptyset\},$$

and let $\mathcal{T}$ *be a minimal* θ*-chain with respect to the pair* (A, α)*.*

(a) Let T *be an element of the set* $\mathcal{T}$ *such that*

$$X \subseteq T \text{ or } T \subseteq X \text{ for all } X \in \mathcal{T}.$$

Let S *be an element of the set* $\mathcal{T}$*. If the set* S *is a proper subset of the set* T*, then the set* S *is a subset of the set*



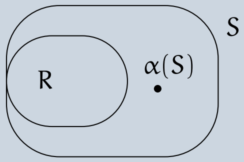
$$T' := T \setminus \{\alpha(T)\},$$

that is, the element $\alpha(T)$ *is not contained in the set* S*.*

(b) Let S *and* T *be two elements of the* θ*-chain* $\mathcal{T}$*. Then we have*

$$S \subseteq T \text{ or } T \subseteq S.$$

(c) Let R *and* S *be two elements of the set* $\mathcal{T}$*. If the set* R *is a proper subset of the set* S*, then the set* R *is also a subset of the set* $S' = S \setminus \{\alpha(S)\}$*.*



Proof. (a) Step 1. *Let*

$$\mathcal{Z}_T := \{Z \in \mathcal{T} \mid Z \subseteq T' \text{ or } T \subseteq Z\}.$$

Then the set $\mathcal{Z}_T$ *is a* θ*-chain:*

We will verify Conditions (i) to (iii) of Definition 10.3:

Condition (i): Since the empty set is a subset of the set T', the empty set is contained in the set $\mathcal{Z}_T$. Since the set T is a subset of the set A, the set A is contained in the set $\mathcal{Z}_T$.

Condition (ii): Let R be an element of the set $\mathcal{Z}_T$ and suppose that $R \neq \emptyset$. We have to show that the set $R' = R \setminus \{\alpha(R)\}$ is also contained in the set $\mathcal{Z}_T$:

Since the set R is contained in the set $\mathcal{Z}_T$, we have

$$R \subseteq T' \text{ or } T \subseteq R.$$

Case 1. *Suppose that the set* R *is a subset of the set* T'*.*

Then we have

$$R' \subseteq R \subseteq T'$$

implying that the set R' is contained in the set $\mathcal{Z}_T$.

Case 2. *Suppose that the set* T *is a subset of the set* R*.*

Since the set R is an element of the set $\mathcal{T}$, the set R' is also an element of the set $\mathcal{T}$ (Condition (ii) of Definition 10.3). By assumption, it follows that

$$R' \subseteq T \text{ or } T \subseteq R'.$$

Case 2.1. *Suppose that the set* R' *is a subset of the set* T.

It follows from

$$R' \subseteq T \subseteq R = R' \cup \{\alpha(R)\}$$

that

$$T = R' \text{ or } T = R.$$

If $T = R'$, then the set T is a subset of the set R' implying that the set R' is contained in the set $\mathcal{Z}_T$.

If $T = R$, it follows that $T' = R'$. In particular, the set R' is a subset of the set T' implying that the set R' is contained in the set $\mathcal{Z}_T$.

Case 2.2. *Suppose that the set* T *is a subset of the set* R'.

Then the set R' is contained in the set $\mathcal{Z}_T$.

Condition (iii): Let $(B_i)_{i \in I}$ be a family of sets contained in the set $\mathcal{Z}_T$. We may distinguish the following two cases:

Case 1. *The set* T *is a subset of the set* B_i *for all elements* i *of the set* I.

It follows that

$$T \subseteq \bigcap_{i \in I} B_i$$

implying that the set $\bigcap_{i \in I} B_i$ is contained in the set $\mathcal{Z}_T$.

Case 2. *There exists an element* j *of the set* I *such that the set* B_j *is contained in the set* T'.

It follows that

$$\bigcap_{i \in I} B_i \subseteq B_j \subseteq T'$$

implying that the set $\bigcap_{i \in I} B_i$ is contained in the set $\mathcal{Z}_T$.

Step 2. *Let* S *be an element of the set* $\mathcal{T}$. *If the set* S *is a proper subset of the set* T, *then the set* S *is also a subset of the set* T':

By Step 1, the set

$$\mathcal{Z}_T = \{Z \in \mathcal{T} \mid Z \subseteq T' \text{ or } T \subseteq Z\}$$

is a θ-chain- Since, by assumption, the set $\mathcal{T}$ is a minimal θ-chain of the set A, it follows that $\mathcal{Z}_T = \mathcal{T}$. Since the set S is an element of the set $\mathcal{T} = \mathcal{Z}_T$, it follows that

$$S \subseteq T' \text{ or } T \subseteq S.$$

If the set S is a subset of the set T', there is nothing to show.

If the set T is a subset of the set S, then we have

$$T \subseteq S \subset T,$$

a contradiction.

(b) Step 1. *Let*
$$\mathcal{Z} := \{Z \in \mathcal{T} \mid X \subseteq Z \text{ or } Z \subseteq X \text{ for all } X \in \mathcal{T}\}.$$
Then the set $\mathcal{Z}$ *is a* θ*-chain:*

We will verity Conditions (i) to (iii) of Definition 10.3:

Condition (i): Since each element of the set $\mathcal{T}$ is a subset of the set A, the set A is contained in the set $\mathcal{Z}$. Since
$$\emptyset \subseteq X \text{ for all } X \in \mathcal{T},$$
the empty set is contained in the set $\mathcal{Z}$.

Condition (ii): Let T be an element of the set $\mathcal{Z}$ such that $T \neq \emptyset$. We have to show that the set $T' = T \setminus \{\alpha(T)\}$ is contained in the set $\mathcal{Z}$. By definition of the set $\mathcal{Z}$, we have
$$X \subseteq T \text{ or } T \subseteq X \text{ for all } X \in \mathcal{T}.$$
Let S be an element of the set $\mathcal{T}$. It follows that we have
$$S \subseteq T \text{ or } T \subseteq S, \text{ that is, } S \subset T \text{ or } T \subseteq S.$$
If the set S is a proper subset of the set T, then it follows from (a) that the set S is also a subset of the set T'.

If the set T is a subset of the set S, then we have
$$T' \subseteq T \subseteq S$$
implying that the set T' is a subset of the set S.

Altogether, we have
$$T' \subseteq S \text{ or } S \subseteq T' \text{ for all } S \in \mathcal{T}$$
implying that the set T' is contained in the set $\mathcal{Z}$.

Condition (iii): Let $(B_i)_{i \in I}$ be a family of sets contained in the set $\mathcal{Z}$, and let X be a set of the θ-chain $\mathcal{T}$. We may distinguish the following two cases:

Case 1. *We have*
$$X \subseteq B_i \text{ for all } i \in I.$$
It follows that
$$X \subseteq \bigcap_{i \in I} B_i.$$
Case 2. *There exists an element* j *of the set* I *such that*
$$B_j \subseteq X.$$
It follows that
$$\bigcap_{i \in I} B_i \subseteq B_j \subseteq X.$$
It follows from Case 1 and Case 2 that we have
$$X \subseteq \bigcap_{i \in I} B_i \text{ or } \bigcap_{i \in I} B_i \subseteq X \text{ for all } X \in \mathcal{T}$$
implying that the set $\bigcap_{i \in I} B_i$ is contained in the set $\mathcal{Z}$.

Step 2. *Let* S *and* T *be two elements of the* θ*-chain* $\mathcal{T}$. *Then we have*

$$S \subseteq T \text{ or } T \subseteq S:$$

Since $\mathcal{T}$ is a minimal θ-set of the set A, it follows from Step 1 that we have

$$\mathcal{T} = \mathcal{Z} = \{Z \in \mathcal{T} \mid Z \subseteq X \text{ or } X \subseteq Z \text{ for all } X \in \mathcal{T}\}.$$

The assertion follows.

(c) Let R and S be two elements of the set $\mathcal{T}$, and suppose that the set R is a proper subset of the set S. Since the set $S' := S \setminus \{\alpha(S)\}$ is also an element of the set $\mathcal{T}$, it follows from (b) that we have

$$S' \subseteq R \text{ or } R \subseteq S'.$$

We want to show that the set R is a subset of the set S'. Hence, we have to consider the case that the set S' is a subset of the set R. It follows from

$$S' = S \setminus \{\alpha(S)\} \subseteq R \subseteq S$$

that $R = S'$ or $R = S$. Since the set R is a proper subset of the set S, it follows that $R = S'$. In particular, the set R is a subset of the set S'. □

We are now able to present the second proof of the well-ordering theorem:

11.3 Theorem. *Every set* A *can be well-ordered, that is, there exists an order* $\leqslant_A$ *on the set* A *such that the pair* $(A, \leqslant_A)$ *is well-ordered.*

Proof. Step 0. The main idea of this second proof of the well-ordering theorem is to reconstruct the set

$$\mathcal{T} := \{T_x \mid x \in A\} \cup \{\emptyset\}$$

of Remark 10.2 as follows:

We will start in Step 1 with an arbitrary choice function

$$\alpha : \mathcal{P}(A) \setminus \{\emptyset\} \to A.$$

By Proposition 11.1, there exists a minimal θ-chain $\mathcal{T}$ of the set A, and we have seen in Proposition 11.2 that we have

$$S \subseteq T \text{ or } T \subseteq S \text{ for all } S,\, T \in \mathcal{T}.$$

In Step 3 and 4 we will define for each element x of the set A the set

$$T_x := \bigcap \{Z \in \mathcal{T} \mid x \in Z\} \text{ for all } x \in A,$$

and we will show that

$$x \in T_x \text{ and } \alpha(T_x) = x.$$

In view of Remark 10.2 we will define in Step 5:

$$x \leqslant y \text{ if and only if } T_y \subseteq T_x \text{ for all } x,\, y \in A.$$

Finally, we will show in Step 6 and 7 that the pair $(A, \leqslant)$ is well-ordered.

Step 1. *Definition of the function $\alpha : \mathcal{P}(A) \to A$:*

By the axiom of choice, there exists a choice function, that is, a function

$$\alpha : \mathcal{P}(A) \to A$$

from the power set $\mathcal{P}(A)$ of the set A into the set A such that

$$\alpha(X) \in X \text{ for all } X \in \mathcal{P}(A) \setminus \{\emptyset\}.$$

Step 2. *Definition of the set $\mathcal{T}$:*

By Proposition 11.1, there exists a minimal θ-chain $\mathcal{T}$ with respect to the pair (A, α).

Step 3. *Definition of the set T_x for an element x of the set A:*

Let x be an element of the set A. Then we set

$$T_x := \bigcap \{Z \in \mathcal{T} \mid x \in Z\},$$

that is, the set T_x is the smallest set of the set $\mathcal{T}$ containing the element x. Note that the set T_x exists since the set A is an element of the set $\mathcal{T}$.

Step 4. *Let x be an element of the set A. Then the set T_x is an element of the set $\mathcal{T}$, and we have*

$$x \in T_x \text{ and } \alpha(T_x) = x :$$

The first assertion follows from the definition of the set T_x and Condition (iii) of Definition 10.3. The second assertion follows directly from the definition of the set T_x.

In order to show that $\alpha(T_x) = x$, assume that $y := \alpha(T_x) \neq x$. Since the set $\mathcal{T}$ is a θ-chain, it follows from Condition (ii) of Definition 10.3 that the set

$$T_x' = T_x \setminus \{\alpha(T_x)\} = T_x \setminus \{y\}$$

is an element of the set $\mathcal{T}$ containing the element x. It follows that

$$T_x' \subset T_x = \bigcap \{Z \in \mathcal{T} \mid x \in Z\} \subseteq T_x',$$

a contradiction.

Step 5. *Definition of the order $\leqslant$ on the set A:*

Let x and y be two elements of the set A. Then we set

$$x \leqslant y \text{ if and only if } T_y \subseteq T_x.$$

Step 6. *The relation $\leqslant$ defined in Step 5 is an order on the set A:*

For, let x, y and z be three elements of the set A.

(i) Since the set T_x is a subset of itself, we have $x \leqslant x$.

(ii) If $x \leqslant y$ and $y \leqslant x$, then we have

$$T_y \subseteq T_x \text{ and } T_x \subseteq T_y$$

implying that $T_x = T_y$. It follows from Step 4 that $x = \alpha(T_x) = \alpha(T_y) = y$.

(iii) If $x \leqslant y$ and $y \leqslant z$, then it follows from

$$T_y \subseteq T_x \text{ and } T_z \subseteq T_y$$

that the set T_z is a subset of the set T_x implying that $x \leqslant z$.

Step 7. *The pair* $(A, \leqslant)$ *is well-ordered:*

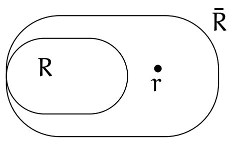
For, let R be a non-empty subset of the set A, let

$$\bar{R} := \bigcap \{Z \in \mathcal{T} \mid R \subseteq Z\}$$

be the smallest set of the set $\mathcal{T}$ containing the set R, and let $r := \alpha(\bar{R})$. We claim that the element r is the minimal element of the set R:

Step 7.1. *The element* r *is contained in the set* R*:*

Assume that the element r is not contained in the set R. Then the set R is a subset of the set



$$\bar{R}' = \bar{R} \setminus \{\alpha(\bar{R})\} = \bar{R} \setminus \{r\}.$$

Since the set $\bar{R}'$ is an element of the set $\mathcal{T}$ (Condition (ii) of Definition 10.3), it follows that

$$\bar{R}' \subset \bar{R} = \bigcap \{Z \in \mathcal{T} \mid R \subseteq Z\} \subseteq \bar{R}',$$

a contradiction.

Step 7.2 *We have* $\bar{R} = T_r$*:*

By Step 7.1, we have

$$T_r = \bigcap \{Z \in \mathcal{T} \mid r \in Z\} \subseteq \bigcap \{Z \in \mathcal{T} \mid R \subseteq Z\} = \bar{R}.$$

Assume that the set T_r is a proper subset of the set $\bar{R}$. Since the sets T_r and $\bar{R}$ are both elements of the set $\mathcal{T}$, it follows from Proposition 11.2 that the set T_r is a subset of the set $\bar{R}'$. It follows that

$$r \in T_r \subseteq \bar{R}' = \bar{R} \setminus \{\alpha(\bar{R})\} = \bar{R} \setminus \{r\},$$

a contradiction.

Step 7.3. *The element* r *is the minimal element of the set* R*:*

For, let x be an element of the set R. Since the element x is contained in the set R, it follows from Step 7.2 that

$$T_x = \bigcap \{Z \in \mathcal{T} \mid x \in Z\} \subseteq \bigcap \{Z \in \mathcal{T} \mid R \subseteq Z\} = \bar{R} = T_r,$$

that is, $r \leqslant x$. □

Historical Notes:

This proof of the well-ordering theorem stems from Zermelo [Zermelo 1908]. It is inspired by the so-called Dedekind chains developed by Richard Dedekind [Dedekind 1888]. The Dedekind chains are explained in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e].

12 Isomorphisms of Well-Ordered Sets

Uniqueness of Isomorphisms between Well-Ordered Sets:

The following proposition is used in the proof of Theorem 12.2.

12.1 Proposition. *Let $\mathrm{A} = (\mathrm{A}, \leqslant_{\mathrm{A}})$ and $\mathrm{B} = (\mathrm{B}, \leqslant_{\mathrm{B}})$ be two non-empty well-ordered sets, and let $\alpha : \mathrm{A} \to \mathrm{B}$ be an isomorphism.*

(a) Let r be the minimal element of the set A, and let s be the minimal element of the set B. Then we have $\alpha(r) = s$.

(b) Let $\alpha : \mathrm{A} \to \mathrm{B}$ and $\beta : \mathrm{A} \to \mathrm{B}$ be two isomorphisms from the set A onto the set B, and let z be an element of the set A. If

$$\alpha(x) = \beta(x) \text{ for all } x \in \mathrm{A}_z = \{x \in \mathrm{A} \mid x < z\},$$

then we have $\alpha(z) = \beta(z)$.

Proof. (a) Since the mapping $\alpha : \mathrm{A} \to \mathrm{B}$ is bijective, there exists an element x of the set A such that $\alpha(x) = s$. It follows from $r \leqslant_{\mathrm{A}} x$ that

$$\alpha(r) \leqslant_{\mathrm{B}} \alpha(x) = s.$$

Since the element s is the minimal element of the set B, it follows that $\alpha(r) = s$.

(b) Assume that $\alpha(z) \neq \beta(z)$. W.l.o.g. we may assume that $\alpha(z) <_{\mathrm{B}} \beta(z)$. Since the mapping $\beta : \mathrm{A} \to \mathrm{B}$ is bijective, there exists an element r of the set A such that $\beta(r) = \alpha(z)$. Since we have

$$\beta(r) = \alpha(z) <_{\mathrm{B}} \beta(z),$$

it follows that $r <_{\mathrm{A}} z$ implying that $\alpha(r) = \beta(r)$. It follows that

$$\alpha(z) = \beta(r) = \alpha(r)$$

implying that $z = r$, in contradiction to $r <_{\mathrm{A}} z$. □

12.2 Theorem. *Let $(\mathrm{A}, \leqslant_{\mathrm{A}})$ and $(\mathrm{B}, \leqslant_{\mathrm{B}})$ be two well-ordered sets, and let $\alpha : \mathrm{A} \to \mathrm{B}$ and $\beta : \mathrm{A} \to \mathrm{B}$ be two isomorphisms from the set A onto the set B. Then we have $\alpha = \beta$.*

Proof. If $\mathrm{A} = \emptyset$, then we have $\mathrm{B} = \emptyset$, and the mapping $\alpha := \emptyset$ is the only isomorphism between the sets A and B.

So we may suppose that $\mathrm{A} \neq \emptyset$ and that $\mathrm{B} \neq \emptyset$. We prove the assertion by transfinite induction:

$\mathrm{A}_z \mapsto z$: Let z be an element of the set A and suppose that we have

$$\alpha(x) = \beta(x) \text{ for all } x \in \mathrm{A}_z = \{x \in \mathrm{A} \mid x <_{\mathrm{A}} z\}.$$

By Proposition 12.1, it follows that $\alpha(z) = \beta(z)$. By transfinite induction, it follows that $\alpha = \beta$. □

12.3 Theorem. *Let $(A, \leqslant)$ be a well-ordered set, and let $\alpha : A \to A$ be an automorphism of the set A. Then we have $\alpha = \mathrm{id}$.*

Proof. The assertion follows from Theorem 12.2. □

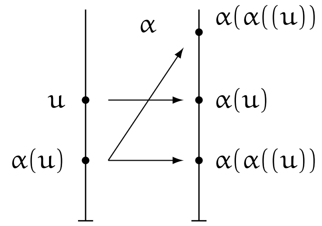
Relation between two Arbitrary Well-Ordered Sets:

12.4 Theorem. *Let $(A, \leqslant)$ be a well-ordered set, and let $\alpha : A \to A$ be an injective homomorphism from the set A into itself. Then we have*

$$x \leqslant \alpha(x) \text{ for all } x \in A.$$

Proof. Let $U := \{x \in A \mid \alpha(x) < x\}$. Assume that $U \neq \emptyset$. Since the set A is well-ordered, there exists a minimal element u of the set U.

Since the element u is contained in the set U, we have $\alpha(u) < u$. Since u is the minimal element of the set U, it follows from $\alpha(u) < u$ that the element $\alpha(u)$ is not contained in the set U implying that $\alpha(\alpha(u)) \geqslant \alpha(u)$.



On the other hand, it follows from $\alpha(u) < u$ that $\alpha(\alpha(u)) \leqslant \alpha(u)$. Together we get

$$\alpha(\alpha(u)) = \alpha(u).$$

Since the mapping $\alpha : A \to A$ is injective it follows that $\alpha(u) = u$, in contradiction to $\alpha(u) < u$. Hence, we have $U = \emptyset$. □

12.5 Remark. Theorem 12.4 can be used for an alternative proof of Theorem 12.2.

Proof. (Second proof of Theorem 12.2) By Proposition 2.4, the functions $\alpha^{-1} : B \to A$ and $\beta^{-1} : B \to A$ are isomorphisms from the set B onto the set A. It follows from Proposition 2.5 that the functions $\beta^{-1} \circ \alpha : A \to A$ and $\alpha^{-1} \circ \beta : A \to A$ are automorphisms of the ordered set A.

Let x be an element of the set A. By Theorem 12.4, we have

$$x \leqslant_A \big(\beta^{-1} \circ \alpha\big)(x) = \beta^{-1}\big(\alpha(x)\big) \text{ and } x \leqslant_A \big(\alpha^{-1} \circ \beta\big)(x) = \alpha^{-1}\big(\beta(x)\big).$$

It follows that $\beta(x) \leqslant_B \alpha(x)$ and $\alpha(x) \leqslant_B \beta(x)$. Hence, we have $\alpha(x) = \beta(x)$. □

12.6 Theorem. *Let $(A, \leqslant)$ be a well-ordered set. Then the set A is not isomorphic to any of its initial segments.*

Proof. Assume that there exist an element a of the set A and an isomorphism $\alpha : A \to A_a$ from the set A onto the initial segment $A_a := \{x \in A \mid x < a\}$. Obviously, the mapping $\alpha : A \to A_a$ extends to an injective mapping $\alpha : A \to A$.

Since the range of α is the initial segment A_a, it follows that the element $\alpha(a)$ is contained in the set A_a implying that $\alpha(a) < a$.

On the other hand, it follows from Theorem 12.4 that $a \leqslant \alpha(a)$, a contradiction. □

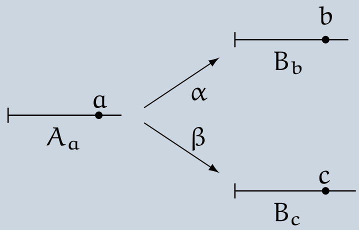
We will need the following propositions in the proof of Theorem 12.10:

12.7 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two well-ordered sets, let* a *be an element of the set* A*, and let*

$$A_a := \{x \in A \mid x <_A a\}$$

be the initial segment of the set A *with respect to the element* a*.*

Suppose that there exist two elements b *and* c *of the set* B *and two isomorphisms* $\alpha : A_a \longrightarrow B_b$ *and* $\beta : A_a \longrightarrow B_c$ *from the initial segment* A_a *onto the initial segments* B_b *and* B_c*, respectively.*



Then we have $b = c$ *and* $\alpha = \beta$*.*

Proof. Assume that $b \neq c$. W.l.o.g. we may assume that $b < c$. Note that the set

$$B_b = \{y \in B \mid y <_B b\} = \{y \in B_c \mid y <_B b\} = \left(B_c\right)_b$$

is an initial segment of the set B_c. Since the mappings $\alpha : A_a \longrightarrow B_b$ and $\beta : A_a \longrightarrow B_c$ are isomorphisms, the mapping

$$\gamma := \alpha \circ \beta^{-1}$$

is an isomorphism from the set B_c onto its initial segment $B_b = \left(B_c\right)_b$. Since the set B_c is well-ordered, this is in contradiction to Theorem 12.6. □

12.8 Proposition. *Let* I *be a totally ordered index set, and let* $(A_i)_{i \in I}$ *and* $(B_i)_{i \in I}$ *be two families of sets with the following properties:*

(i) The families $(A_i)_{i \in I}$ *and* $(B_i)_{i \in I}$ *are chains, that is, we have*

$$A_i \subseteq A_j \text{ and } B_i \subseteq B_j \text{ if } i \leqslant j.$$

(ii) For each element i *of the set* I*, there exists a function* $\alpha_i : A_i \longrightarrow B_i$ *from the set* A_i *into the set* B_i*.*

(iii) For each two elements i *and* j *of the set* I *such that* $i \leqslant j$*, the function* $\alpha_i : A_i \longrightarrow B_i$ *is induced by the function* $\alpha_j : A_j \longrightarrow B_j$*, that is, we have*

$$\alpha_i(x) = \alpha_j(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i \in I} A_i \text{ and } B := \bigcup_{i \in I} B_i.$$

(a) There exists exactly one function $\alpha : A \longrightarrow B$ *from the set* $A = \bigcup_{i \in I} A_i$ *into the set* $B = \bigcup_{i \in I} B_i$ *such that* $\alpha|_{A_i} = \alpha_i$ *for all elements* i *of the set* I*.*

(b) If the functions $\alpha_i : A_i \longrightarrow B_i$ *are bijective for all elements* i *of the set* I*, then the function* $\alpha : A \longrightarrow B$ *is also bijective.*

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c]. □

12.9 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two well-ordered sets, and let* a *and* b *be two elements of the sets* A *and* B*, respectively. Suppose that there exists an isomorphism* $\alpha : A_a \to B_b$ *from the initial segment* A_a *onto the initial segment* B_b*.*

Let

$$\bar{A}_a := \{x \in A \mid x \leqslant_A a\} \text{ and } \bar{B}_b := \{x \in B \mid x \leqslant_B b\},$$

and define the function $\bar{\alpha} : \bar{A}_a \to \bar{B}_b$ *by*

$$\bar{\alpha}(x) := \begin{cases} \alpha(x) & \text{if } \; x \in A_a = \{x \in A \mid x <_A a\} \\ b & \text{if } \; x = a. \end{cases}$$

Then the function $\bar{\alpha} : \bar{A}_a \to \bar{B}_b$ *is an isomorphism from the well-ordered set* $\bar{A}_a$ *onto the well-ordered set* $\bar{B}_b$*.*

Proof. By Proposition 5.10, the sets $\bar{A}_a$ and $\bar{B}_b$ are well-ordered. Since the function $\alpha : A_a \to B_b$ is an isomorphism, it follows that the function $\bar{\alpha} : \bar{A}_a \to \bar{B}_b$ is an isomorphism, as well. □

12.10 Theorem. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two well-ordered sets. Then one of the following cases occurs:*

(i) The sets A *and* B *are isomorphic.*

(ii) There exists an element a *of the set* A *such that the set* B *is isomorphic to the initial segment* $A_a := \{x \in A \mid x <_A a\}$ *of the set* A*.*

(iii) There exists an element b *of the set* B *such that the set* A *is isomorphic to the initial segment* $B_b := \{y \in B \mid y <_B b\}$ *of the set* B*.*

Proof. Step 1. *Let*

$$C := \{x \in A \mid \exists\, x' \in B \;\exists\, \alpha_x : A_x \to B_{x'} \text{ isomorphism}\}$$

be the set of the elements x *of the set* A *such that there exists an element* x' *of the set* B *and an isomorphism* $\alpha_x : A_x \to B_{x'}$ *from the initial segment* A_x *onto the initial segment* $B_{x'}$*. Then the following conditions are fulfilled:*

(i) We have

$$A_x \subseteq A_y \textit{ for all } x, y \in C \textit{ such that } x \leqslant y.$$

(ii) We have

$$B_{x'} \subseteq B_{y'} \textit{ for all } x, y \in C \textit{ such that } x \leqslant y.$$

(iii) We have

$$\alpha_x = \alpha_y|_{A_x} \textit{ for all } x, y \in C \textit{ such that } x \leqslant y :$$

(i) Obviously, we have

$$A_x = \{z \in A \mid z < x\} \subseteq \{z \in A \mid z < y\} = A_y \text{ for all } x, y \in C \text{ such that } x \leqslant y.$$

(iii) Let x and y be two elements of the set C such that $x \leqslant y$, and set $\beta_x := \alpha_y|_{A_x}$. Note that there exists an element b of the set B such that $\alpha_y(A_x) = B_b$. It follows that the mappings

$$\alpha_x : A_x \to B_{x'} \text{ and } \beta_x : A_x \to \alpha_y(A_x) = B_b$$

are isomorphisms from the initial segment A_x onto the initial segments $B_{x'}$ and $\alpha_y(A_x) = B_b$, respectively. By Proposition 12.7, we have

$$x' = b,\ B_{x'} = B_b = \alpha_y(A_x) \text{ and } \alpha_x = \beta_x = \alpha_y|_{A_x}.$$

(ii) If $x \leqslant y$, then it follows from (iii) that

$$B_{x'} = \alpha_y(A_x) \subseteq \alpha_y(A_y) = B_{y'}.$$

Step 2. *Let*

$$D := \bigcup_{z \in C} A_z \textit{ and } R := \bigcup_{z \in C} B_{z'}.$$

Then the following conditions are fulfilled:

(i) We have $D = A$, or the set D is an initial segment of the set A.

(ii) We have $R = B$, or the set R is an initial segment of the set B.

(iii) There exists an isomorphism $\alpha : D \to R$ from the set D onto the set R such that

$$\alpha_x = \alpha|_{A_x} \textit{ for all } x \in C:$$

(i) and (ii) Conditions (i) and (ii) follow from Theorem 7.4.

(iii) By Proposition 12.8, there exists a bijective function $\alpha : D \to R$ from the set D onto the set R such that

$$\alpha_x = \alpha|_{A_x} \text{ for all } x \in C.$$

Let x and y be two elements of the set D. Since $D = \bigcup_{z \in C} A_z$, there exists an element r of the set C such that the elements x and y are contained in the set A_r. It follows that there exists an element r' of the set B and an isomorphism $\alpha_r : A_r \to B_{r'}$ from the initial segment A_r onto the initial segment $B_{r'}$.

Since $\alpha_r = \alpha|_{A_r}$ and since the mapping $\alpha_r : A_r \to B_{r'}$ is an isomorphism from the initial segment A_r onto the initial segment $B_{r'}$, it follows that

$$x \leqslant y \Leftrightarrow \alpha_r(x) \leqslant \alpha_r(y) \Leftrightarrow \alpha(x) \leqslant \alpha(y).$$

Hence, the mapping $\alpha : D \to R$ is an isomorphism from the set D onto the set R.

Step 3. *One of the following cases occurs:*

(i) The sets A and B are isomorphic.

(ii) There exists an element c of the set A such that the set B is isomorphic to the initial segment $A_c := \{x \in A \mid x < c\}$.

(iii) There exists an element d of the set B such that the set A is isomorphic to the initial segment $B_d := \{y \in B \mid y < d\}$:

By Step 2, we have $D = A$, or there exists an element a of the set A such that $D = A_a := \{x \in A \mid x < a\}$.

By Step 2, we have $R = B$, or there exists an element b of the set B such that $R = B_b := \{y \in B \mid y < b\}$.

By Step 2, the function $\alpha : D \to R$ is an isomorphism.

If $D = A$ and $R = B$, then we obtain an isomorphism $\alpha : A \to B$, that is, $A \cong B$ (Case (i)).

If $D = A_a$ and $R = B$, then we obtain an isomorphism $\alpha : A_a \to B$, that is, $A_a \cong B$ (Case (ii) with $c = a$).

If $D = A$ and $R = B_b$, then we obtain an isomorphism $\alpha : A \to B_b$, that is, $A \cong B_b$ (Case (iii) with $d = b$).

It remains to consider the case where $D = A_a$ and $R = B_b$. In this case we have an isomorphism $\alpha : A_a \to B_b$. It follows from Proposition 12.9 that there exists an isomorphism

$$\bar{\alpha} : \bar{A}_a := \{x \in A \mid x \leqslant a\} \to \bar{B}_b := \{y \in B \mid y \leqslant b\}$$

such that $\bar{\alpha}(x) = \alpha(x)$ for all elements x of the set A_a and such that $\bar{\alpha}(a) = b$.

By Theorem 7.2, we have $\bar{A}_a = A$ or there exists an element c of the set A such that $\bar{A}_a = A_c$, and we have $\bar{B}_b = B$ or there exists an element d of the set B such that $\bar{B}_b = B_d$.

If $\bar{A}_a = A$ and $\bar{B}_b = B$, then we obtain an isomorphism $\bar{\alpha} : A \to B$, that is, $A \cong B$ (Case (i)).

If $\bar{A}_a = A_c$ and $\bar{B}_b = B$, then we obtain an isomorphism $\bar{\alpha} : A_c \to B$, that is, $A_c \cong B$ (Case (ii)).

If $\bar{A}_a = A$ and $\bar{B}_b = B_d$, then we obtain an isomorphism $\bar{\alpha} : A \to B_d$, that is, $A \cong B_d$ (Case (iii)).

It remains to consider the case $\bar{A}_a = A_c$ and $\bar{B}_b = B_d$. It follows that the element c is contained in the set C defined in Step 1. Hence, we have

$$a \in \bar{A}_a = A_c \subseteq \bigcup_{z \in C} A_z = D = A_a,$$

a contradiction. This contradiction means that the case $\bar{A}_a = A_c$ and $\bar{B}_b = B_d$ cannot occur. □

12.11 Remark. Note that Theorem 12.10 is a very strong result about well-ordered sets: It says that two arbitrary well-ordered sets are either isomorphic, or one of them is isomorphic to an initial segment of the other one.

Historical Notes:

Theorem 12.2 is due to Cantor:

B. *Zwei ähnliche wohlgeordnete Mengen F und G lassen sich nur auf eine einzige Weise aufeinander abbilden.*

See [Cantor 1897, p. 212].

B. *Given two isomorphic well-ordered sets F and G there is only one isomorphism from one set onto the other.*

Theorem 12.4 is due to Zermelo:

I did not find an article of Zermelo where this result is published but a hint in the collected work of Cantor edited by Zermelo:

Hier lassen sich aber die Sätze B - F einfacher als bei Cantor beweisen bzw. ersetzen durch Voranstellung des allgemeinen vom Herausgeber [Zermelo] herrührenden Hilfssatzes: Bei keiner ähnlichen Abbildung einer wohlgeordneten Menge auf einen ihrer Teile wird ein Element a auf ein vorangehendes $a' < a$ abgebildet.

See [Cantor 1932, p. 354].

Here, however, Theorems B - F can be proven more easily than with Cantor or even replaced by placing the general auxiliary proposition from the editor [Zermelo] in front: An isomorphism from a well-ordered set never sends an element a *to a preceding element* $a' < a$.

Probably, Zermelo even proved the stronger result that a monomorphism from a well-ordered set never sends an element a to a preceding element $a' < a$.

Theorem 12.6 is due to Cantor:

B. *Eine wohlgeordnete Menge ist keinem ihrer Abschnitte ähnlich.*

See [Cantor 1897, p. 211].

B. *A well-ordered set is not isomorphic to any of its initial segments.*

Proposition 12.7 is due to Cantor:

F. *Sind* F *und* G *zwei wohlgeordnete Mengen, so kann ein Abschnitt* A *von* F *höchstens einen ihm ähnlichen Abschnitt* B *in* G *haben.*

See [Cantor 1897, p. 213].

B. *If* F *and* G *are two well-ordered sets, and if* A *is an initial segment of* F*, then there exists at most one initial segment in* G *isomorphic to* A.

Theorem 12.10 is due to Cantor:

N. *Sind* F *und* G *zwei beliebige wohlgeordnete Mengen, so sind entweder 1)* F *und* G *einander ähnlich, oder es gibt 2) einen bestimmten Abschnitt* B_1 *von* G*, welcher* F *ähnlich ist, oder es gibt 3) einen bestimmten Abschnitt* A_1 *von* F*, welcher* G *ähnlich ist, und jeder dieser drei Fälle schließt die Möglichkeit der beiden anderen aus.*

See [Cantor 1897, p. 215].

B. *If* F *and* G *are two well-ordered sets, then either 1) the sets* F *and* G *are isomorphic or 2) there exists an initial segment* B_1 *isomorphic to* F *or 3) there exists an initial segment* A_1 *isomorphic to* G*. Each of these three possibilities excludes the two others.*

13 The Equivalence of the Axiom of Choice, the Lemma of Zorn and the Well-Ordering Theorem

In the walk *The Axioms of Zermelo and Fraenkel* we have introduced the axioms of Zermelo and Fraenkel as follows:

ZFC-0:	Basic Axiom
ZFC-1:	Axiom of Extension
ZFC-2:	Axiom of Existence
ZFC-3:	Axiom of Specification
ZFC-4:	Axiom of Pairing
ZFC-5:	Axiom of Unions
ZFC-6:	Axiom of Powers
ZFC-7:	Axiom of Foundation
ZFC-8:	Axiom of Substitution
ZFC-9:	Axiom of Choice
ZFC-10:	Axiom of Infinity

In Theorem 13.1 we will show that we can replace the Axiom of Choice (ZFC-9) by the Lemma of Zorn or by the well-ordering Theorem. Note that the term *mathematical universe* is defined in Unit *The Mathematical Universe* [Garden 2020a].

13.1 Theorem. *Let $\mathfrak{U}$ be a mathematical universe fulfilling the axioms ZFC-1 to ZFC-8 and ZFC-10. Then the following statements are equivalent:*

(i) The Axiom of Choice (ZFC-9).

(ii) The Lemma of Zorn (Theorem 9.6).

(iii) The well-ordering Theorem (Theorem 9.7).

Proof. (i) $\Rightarrow$ (ii): For a proof of the Lemma of Zorn see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020c].

(ii) $\Rightarrow$ (iii): It has been shown in Theorem 9.7 that the well-ordering Theorem is a consequence of the Lemma of Zorn.

(i) $\Rightarrow$ (iii): It has been shown in Theorem 11.3 that the well-ordering Theorem is a consequence of the Axiom of Choice.

(iii) $\Rightarrow$ (i): Let $(A_i)_{i \in I}$ be a family of non-empty sets, and let

$$A := \bigcup_{i \in I} A_i.$$

By the well-ordering Theorem (Theorem 9.7), there exists a well-ordering $\leqslant$ on the set A. It follows that for each element i of the set I there exists a minimal element x_i of the set A_i. Let $f : I \to A$ be defined by $f : i \mapsto x_i$.

Then the function $f : I \to A$ is an element of the set $\prod_{i \in I} A_i$. In particular, the set $\prod_{i \in I} A_i$ in non-empty. $\square$

14 Notes and References

Do you want to learn more? The next unit of the walk *The Cardinality of Sets*, Unit *Ordinal Numbers* [Garden 2020g], is devoted to the study of ordinal numbers. They are an extension of the natural numbers and allow to “count” the number of the elements of a well-ordered set.

15 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Cantor, Georg (1874). “Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen”. In: *Journal für die reine und angewandte Mathematik* 77, pp. 258–262. See DigiZeitschriften (cit. on p. 3).

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Cohen, Paul Joseph (1963). "The Independence of the Continuum Hypothesis". In: *Proceedings of the National Academy of Sciences of the United States of America* 50.6, pp. 1143–1148. See National Academy (cit. on p. 31).

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Zorn, Max (1935). "A remark on Method in Transfinite Algebra". In: *Bulletin of the American Mathematical Society* 41.10, pp. 667–670. See Project Euclid (cit. on p. 31).

16 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *The Mathematical Universe*. URL: `https://www.math-garden.com/unit/nst-universe` (cit. on p. 49).

— (2020b). *Families and the Axiom of Choice*. URL: `https://www.math-garden.com/unit/nst-families` (cit. on p. 33).

— (2020c). *Ordered Sets and the Lemma of Zorn*. URL: `https://www.math-garden.com/unit/nst-ordered-sets` (cit. on pp. 9–11, 15, 25–27, 44, 49).

— (2020d). *Successor Sets and the Axioms of Peano*. URL: `https://www.math-garden.com/unit/nst-successor-sets` (cit. on p. 12).

— (2020e). *The Natural Numbers and the Principle of Induction*. URL: `https://www.math-garden.com/unit/nst-natural-numbers` (cit. on p. 41).

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— (2020g). *Ordinal Numbers*. URL: `https://www.math-garden.com/unit/card-ordinal` (cit. on pp. 3, 6, 7, 23, 25, 49).

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Index