M. Garden

Ordinal Numbers

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Contents

1 Introduction

The present unit is part of the walk *The Cardinality of Sets* consisting of the following units:

1. Finite Sets and their Cardinalities [Garden 2020g]
2. Well-Ordered Sets [Garden 2020h]
3. Ordinal Numbers (this unit)
4. Cardinal Numbers [Garden 2020i]
5. Cardinal Arithmetic [Garden 2020j]
6. The Axiomatics of von Neumann, Bernays and Gödel [Garden 2020k]

*

One important property of the natural numbers is that they provide the possibility to count the number of the elements of a finite set: A set A is finite with $n+1$ elements if there exists a natural number n and a bijective mapping $\alpha : A \to \{1, \ldots, n+1\}$ from the set A onto the set $\{1, \ldots, n+1\}$ or equivalently, a bijective mapping $\alpha : A \to \{0, 1, \ldots, n\}$ from the set A onto the set $\{0, \ldots, n\}$. In this case we write $|A| = n+1$.

Since

$$n+1 = \{0, 1, \ldots, n\},$$

we can also say that there exists a bijective mapping from the set A onto the set $n+1$.

In this unit we want to transfer this approach from the natural numbers to the so-called *ordinal numbers* where the ordinal numbers shall be used to "count" the number of elements of a well-ordered set. For well-ordered sets see Unit *Well-Ordered Sets* [Garden 2020h].

The idea is first to give a formal definition of an ordinal number and then to show that for each well-ordered set $A = (A, \leqslant_A)$ there exists exactly one ordinal number $N = (N, \leqslant_N)$ isomorphic to the set A.

There is an important point to notice: We do not only assume the existence of a bijective mapping between the sets A and N but of an isomorphism between the ordered sets $(A, \leqslant_A)$ and $(N, \leqslant_N)$. So an ordinal number is not really the cardinality of a set, but a sort of cardinality of a well-ordered set. We will come back to the concept of cardinality in Unit *Cardinal Numbers* [Garden 2020i].

So let us look at the way how ordinal numbers are defined: Ordinal numbers are an invention of Georg Cantor. However, the modern definition of ordinal numbers is due to John von Neumann. The principal idea is to start with the natural numbers and then to continue in the same way: So the first ordinal numbers are just the natural numbers which are successively defined as follows:

$$0 := \emptyset,\ 1 := 0 \cup \{0\} = \{0\},\ 2 := 1 \cup \{1\} = \{0, 1\}, \ldots, n+1 := n \cup \{n\} = \{0, 1, \ldots, n\}, \ldots$$

The natural numbers stop here, but the ordinal numbers continue by defining the next ordinal number N by

$$N := \bigcup_{n \in \mathbb{N}_0} n = \mathbb{N}_0.$$

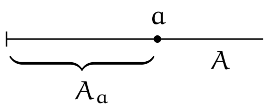
So the ordinal number N is simply the set $\mathbb{N}_0$ of all natural numbers. Now we can continue as before: The next ordinal number N' is simply the set

$$N' := N \cup \{N\} = \mathbb{N}_0 \cup \{\mathbb{N}_0\}$$

and so on. Well, the "and so on" is not so simple since it is not clear how to give a formal description of this "algorithm". John von Neumann had the ingenious idea to define the ordinal numbers by a simple property and to show that they are exactly as we want them to be. His definition makes use of the initial segment of an ordered set:

Let $A = (A, \leqslant)$ be an ordered set, and let a be an element of the set A. Then the set

$$A_a := \{x \in A \mid x < a\}$$



is called **the initial segment of the set A with respect to the element a.**

If we look at a natural number $n+1$ or at the set $\mathbb{N}_0$ of the natural numbers, then we may notice that they have a quite strange property: Let A be the natural number $n+1$ or the set $\mathbb{N}_0$ of the natural numbers. Then we have

$$a = A_a \text{ for all } a \in A.$$

In other words, each element a of the set A is at the same time the initial segment A_a of the set A with respect to this element a. This simple observation follows from the fact that each natural number m fulfills the equation

$$\begin{aligned} m+1 &= \{0, 1, \ldots, m\} = \{x \in \mathbb{N}_0 \mid x < m+1\} = (\mathbb{N}_0)_{m+1} \\ &= \{x \in n+1 \mid x < m+1\} = (n+1)_{m+1} \text{ if } m < n. \end{aligned}$$

In Unit *Well-Ordered Sets* [Garden 2020h] we have explained why only specific orderings, the so-called well-orderings, are suitable orderings for counting. We bring these two elements together and define an ordinal number as follows:

An ordered set $A = (A, \leqslant)$ is called an **ordinal number** if the set A is well-ordered and if any element a of the set A fulfills the equation $a = A_a$ (see Definition 3.2).

It turns out that for each well-ordered set $A = (A, \leqslant)$ there exists exactly one ordinal number $N = (N, \leqslant)$ such that the ordered sets A and N are isomorphic (see Theorem 7.7).

It is time to come back to the initial question of generalizing the concept of cardinality from finite sets to infinite sets. We want to develop a concept of *cardinal numbers* with the property that for each set A there exists exactly one cardinal number C such that there exists a bijective mapping between the sets A and C.

The ordinal numbers are too fine for that approach: By the well-ordering theorem (Theorem 7.1), each set A can be endowed with a well-ordering $\leqslant_A$. Since each well-ordered set is isomorphic to an ordinal number $N = (N, \leqslant_N)$, we get the mapping

$$A \mapsto (A, \leqslant_A) \mapsto (N, \leqslant_N)$$

However, one may endow a set A with different well-orderings which yield different ordinal numbers. An example is explained in Remark 7.8. We will explain the concept of cardinal numbers in Unit *Cardinal Numbers* [Garden 2020i].

The material of the present unit is organized as follows:

Background (see Section 2):

We recall some results about ordered sets, isomorphisms between ordered sets, successor sets and natural numbers.

Definition of Ordinal Numbers (see Section 3):

An ordered set $A = (A, \leqslant)$ is called an **ordinal number** if the pair $A = (A, \leqslant)$ is well-ordered and if we have

$$a = A_a = \{x \in A \mid x < a\} \text{ for all } a \in A,$$

that is, if each element a of the well-ordered set A equals its initial segment (Definition 3.2).

We will study in Section 3 the elementary properties of ordinal numbers. The most important elementary properties are:

Each initial segment of an ordinal number is again an ordinal number (Proposition 3.4).

If $A = (A, \leqslant)$ is an ordinal number, then the one point extension $A^+ := A \cup \{A\}$ is also an ordinal number (Theorem 3.7).

An elementary consequence of the definition of ordinal numbers is the fact that ordinal numbers are transitive sets, that is, each element a of an ordinal number A is also a subset of the set A (Proposition 3.9).

As explained above the pair $(\mathbb{N}_0, \leqslant)$ of the natural numbers with the standard order is an ordinal number (Theorem 3.11), and each natural number n, more precisely, each pair $(n, \leqslant)$ with the standard order is an ordinal number (Theorem 3.12).

Isomorphisms between Ordinal Numbers (see Section 4):

In Section 4 we will study the isomorphisms between ordinal numbers. It will turn out that ordinal numbers are in the following sense “unique”: If $A = (A, \leqslant_A)$ and $B = (B, \leqslant_B)$ are two isomorphic ordinal numbers, then we have $(A, \leqslant_A) = (B, \leqslant_B)$, and every isomorphism $\alpha : A \to B$ from the ordinal number A onto the ordinal number B is the identity map (Theorem 4.2).

As a corollary we obtain the result that the automorphism group $\mathrm{Aut}(A)$ of an ordinal number is trivial, that is, $\mathrm{Aut}(A) = \{\mathrm{id}\}$ (Proposition 4.4).

As a further consequence of Theorem 4.2 we will show in Theorem 4.6 that each two ordinal numbers are closely related: If $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are two ordinal numbers, then exactly one of the following cases occurs:

(i) $A = B$ or

(ii) $A = b = B_b$ for an element b of the set B (in particular $A \in B$ and $A \subset B$) or

(iii) $B = a = A_a$ for an element a of the set A (in particular $B \in A$ and $B \subset A$).

A first consequence of Theorem 4.6 is the following property of ordinal numbers: If A and B are two ordinal numbers, then we have

$$A \in B \text{ if and only if } A \subset B \text{ (Theorem 4.9).}$$

A second consequence of Theorem 4.6 is that we have

$$A \subseteq B \text{ or } B \subseteq A$$

for all ordinal numbers A and B.

The Standard Order on the Ordinal Numbers (see Section 5):

The property that $A \subseteq B$ or $B \subseteq A$ for all ordinal numbers A and B will be used in Section 5 to define a standard (total) order on the ordinal numbers by setting

$$A \leqslant B \text{ if and only if } A \subseteq B \text{ (Definition 5.1).}$$

At a first glance one might misunderstand the above definition by supposing that the *set* of the ordinal numbers can be endowed with an order. But it will turn out in Theorem 6.3 (Theorem of Burali-Forti) that the ordinal numbers do not form a set.

Therefore, we have to be a little more restrictive by saying that for each set $\mathcal{A}$ of ordinal numbers the above definition of the standard order provides a totally ordered set $(\mathcal{A}, \leqslant)$ (Theorem 5.3).

Even more, we will be able to show that each set of ordinal numbers is well-ordered (Theorem 5.6).

The Theorem of Burali-Forti (see Section 6):

In Section 6 we will prove the Theorem of Burali-Forti mentioned above (Theorem 6.3). The main tool in the proof of the Theorem of Burali-Forti is the following result:

Let $\mathcal{A}$ be a set of ordinal numbers. Then the set

$$S := \bigcup_{A \in \mathcal{A}} A$$

is also an ordinal number. In addition, if T is an ordinal number such that

$$A \leqslant T \text{ for all } A \in \mathcal{A},$$

then we have $S \leqslant T$ (Theorem 6.1).

Well-Ordered Sets and Ordinal Numbers (see Section 7):

In the last section of this unit we will explain the relation between well-ordered sets and ordinal numbers. The main result of this section says that every well-ordered set is isomorphic to an ordinal number (Theorem 7.7).

We will prove Theorem 7.7 by transfinite induction. The main tool for the proof is Proposition 7.6 saying that a well-ordered set A with the property that each initial segment A_x is isomorphic to an ordinal number is itself isomorphic to an ordinal number.

2 Background

The discussion of ordinal numbers in this unit requires some background explained in other units. The necessary background is summarized in this section. For more information about ordered sets and initial segments see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d], about successor sets and transitive sets see Unit *Successor Sets and the Axioms of Peano* [Garden 2020e], about natural numbers see Unit *Natural Numbers and the Principle of Induction* [Garden 2020f] and about well-ordered sets see Unit *Well-Ordered Sets* [Garden 2020h].

Ordered Sets:

2.1 Definition. Let A be a set.

(a) A relation $\leqslant$ on the set A is called an **order** or, equivalently, a **partial order on the set** A if it fulfills the following conditions:

(i) The relation $\leqslant$ is **reflexive**, that is, $x \leqslant x$ for all elements x of the set A.

(ii) The relation $\leqslant$ is **antisymmetric**, that is, $x \leqslant y$ and $y \leqslant x$ imply $x = y$ for all elements x and y of the set A.

(iii) The relation $\leqslant$ is **transitive**, that is, $x \leqslant y$ and $y \leqslant z$ imply $x \leqslant z$ for all elements x, y and z of the set A.

(b) A set A with a (partial) order $\leqslant$ is called an **ordered set** or, equivalently, a **partially ordered set** and is often denoted by $\mathcal{A} = (A, \leqslant)$.

(c) A (partial) order $\leqslant$ on a set A is called a **total order** if we have $x \leqslant y$ or $y \leqslant x$ for all elements x and y of the set A. A set A with a total order $\leqslant$ is called a **totally ordered set**.

2.2 Definition. Let $\mathcal{A} = (A, \leqslant_A)$ be an ordered set, let B be a subset of the set A, and let $\leqslant_B$ be the order on the set B such that

$$x \leqslant_B y \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in B.$$

The order $\leqslant_B$ is called **the order on the set B induced by the order $\leqslant_A$**.

Often we write $\mathcal{A} = (A, \leqslant)$ and $\mathcal{B} = (B, \leqslant)$ without explicitly distinguishing between the order $\leqslant_A$ on the set A and the induced order $\leqslant_B$ on the set B.

Minimal and Maximal Elements:

2.3 Definition. Let $\mathcal{A} = (A, \leqslant)$ be an ordered set, and let B be a subset of the set A.

(a) An element b of the set B is called a **minimal element** or a **minimum** of the set B if we have $x \geqslant b$ for all elements x of the set B.

(b) An element b of the set B is called a **maximal element** or a **maximum** of the set B if we have $x \leqslant b$ for all elements x of the set B.

2.4 Proposition. *Let* $A = (A, \leqslant)$ *be a totally ordered set, and let* B *be a subset of the set* A.

(a) If the set B *admits a maximum, then the maximum is unique.*

(b) If the set B *admits a minimum, then the minimum is unique.*

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

2.5 Definition. Let $A = (A, \leqslant_A)$ be an ordered set, and let b be an element not contained in the set A. Set $B := A \cup \{b\}$. We define an order $\leqslant_B$ on the set B as follows:

For each two elements x and y of the set A, set $x \leqslant_B y$ if and only if $x \leqslant_A y$. For each element x of the set B, set $x \leqslant_B b$.

The pair $(B, \leqslant_B)$ is called the **one point extension** of the pair $(A, \leqslant_A)$.

Isomorphisms of Ordered Sets:

2.6 Definition. Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \to B$ be a function from the set A into the set B.

(a) The function $\alpha : A \to B$ is called a **homomorphism of ordered sets** if we have

$$\alpha(x) \leqslant_B \alpha(y) \text{ for all } x, y \in A \text{ such that all } x \leqslant_A y.$$

A homomorphism of ordered sets is also called a **homomorphism** if no danger of confusion arises.

(b) A bijective homomorphism $\alpha : A \to B$ with the property that the inverse function $\alpha^{-1} : B \to A$ is also a homomorphism is called an **isomorphism of ordered sets**.

(c) The ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are called **isomorphic** if there exists an isomorphism $\alpha : A \to B$ from the set A onto the set B. If the sets A and B are isomorphic, then we write $A \cong B$.

2.7 Proposition. *Let* $(A, \leqslant_A)$, $(B, \leqslant_B)$ *and* $(C, \leqslant_C)$ *be three ordered sets.*

(a) Suppose that there exist two isomorphisms $\alpha : A \to B$ *and* $\beta : B \to C$ *from the set* A *onto the set* B *and from the set* B *onto the set* C*, respectively.*

Then the composite $\gamma := \beta \circ \alpha : A \to C$ *is an isomorphism from the set* A *onto the set* C.

(b) If $A \cong B$ *and* $B \cong C$*, then we have* $A \cong C$.

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

Initial Segments:

2.8 Definition. Let $(A, \leqslant)$ be an ordered set, and let a be an element of the set A. Then the set

$$A_a := \{x \in A \mid x < a\}$$

is called the **initial segment of the set A with respect to the element a**.

2.9 Proposition. *Let A be an ordered set, let a be an element of the set A, and let $A_a := \{x \in A \mid x < a\}$ be the initial segment of the set A with respect to the element a. Let z be an element of the initial segment A_a. Then we have*

$$\left(A_a\right)_z = A_z$$

where the sets A_z and $\left(A_a\right)_z$ denote the initial segments of the sets A and A_a with respect to the element z, respectively.

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

2.10 Proposition. *Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \to B$ be an isomorphism from the ordered set A onto the ordered set B.*

Let a be an element of the set A, and let $A_a := \{x \in A \mid x < a\}$ be the initial segment of the set A with respect to the element a. Then the set $\alpha(A_a)$ is the initial segment of the set B with respect to the element $b := \alpha(a)$, that is,

$$\alpha(A_a) = B_b = B_{\alpha(a)}.$$

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

Successor Sets:

2.11 Definition. Let A be a set. Then the set $A^+ := A \cup \{A\}$ is called the **successor** of the set A.

2.12 Proposition. *Let A be a set, and let A^+ be its successor. Then the successor A^+ is non-empty, and we have $A^+ \neq A$.*

Proof. See Unit *Successor Sets and the Axioms of Peano* [Garden 2020e]. □

Natural Numbers:

In Unit *Natural Numbers and the Principle of Induction* [Garden 2020f] we have introduced the natural numbers as specific sets, namely

$$0 := \emptyset, \; 1 := \{0\}, \; 2 := \{0, 1\}, \; \ldots$$

and so on. More generally, we have:

2.13 Theorem. *Let n be a natural number.*

(a) We have $n + 1 = \{0, 1, 2, \dots, n\}$.

(b) For each natural number n, the set $n + 1 := n \cup \{n\}$ is a natural number.

(c) We have $(\mathbb{N}_0)_n := \{x \in \mathbb{N}_0 \mid x < n\} = n$ (initial segment).

Proof. See Unit *Natural Numbers and the Principle of Induction* [Garden 2020f]. □

3 Definition of Ordinal Numbers

We will define ordinal numbers (Definition 3.2), and we will explain their elementary properties. In addition, we will see that each natural number is an ordinal number (Theorem 3.12) and that the set $\mathbb{N}_0$ of the natural numbers is an ordinal number (Theorem 3.11).

Definition of Ordinal Numbers:

Ordinal numbers are specific well-ordered sets. We start by recalling the definition of a well-ordered set.

3.1 Definition. Let $(A, \leqslant)$ be a totally ordered set.

(a) The pair $(A, \leqslant)$ is called **well-ordered** if every non-empty subset X of the set A has a minimal element, that is, if for every subset X of the set A there exists an element z of the set X such that

$$z \leqslant x \text{ for all } x \in X.$$

(b) If the pair $(A, \leqslant)$ is well-ordered, then the order $\leqslant$ is called a **well-ordering**.

For more details about well-ordered sets see Union *Well-Ordered Sets* [Garden 2020h].

3.2 Definition. Let $A = (A, \leqslant)$ be a well-ordered set. The pair $(A, \leqslant)$ is called an **ordinal number** if all elements a of the set A fulfill the following condition:

$$a = A_a := \{x \in A \mid x < a\},$$

that is, each element a of the set A equals the initial segment A_a of the set A with respect to the element a.

French / German. Ordinal Number = Nombre Ordinal = Ordinalzahl.

Elementary Properties of Ordinal Numbers:

We will need the following Proposition about well-ordered sets in the proof of Proposition 3.4

3.3 Proposition. *Let $(A, \leqslant)$ be a well-ordered set, and let B be a subset of the set A. Then the set $B = (B, \leqslant)$ (induced order) is also well-ordered.*

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

3.4 Proposition. *Let $A = (A, \leqslant)$ be an ordinal number, and let a be an element of the set A. Then the initial segment $A_a := \{x \in A \mid x < a\}$ is also an ordinal number.*

Proof. Let a be an element of the set A. By Proposition 3.3, the initial segment A_a equipped with the order induced by the order of the set A is also well-ordered. Since A is a totally ordered set, it follows from Proposition 2.9 that

$$(A_a)_z = A_z = z$$

where $(A_a)_z$ denotes the initial segment of the set A_a with respect to the element z. □

3.5 Proposition. *Let $(A, \leqslant_A)$ be an ordinal number. Then every element of the set A is again an ordinal number.*

Proof. Let a be an element of the set A. By definition of an ordinal number, we have

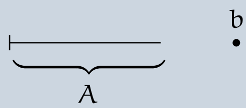
$$a = A_a = \{x \in A \mid x < a\}.$$

By Proposition 3.4, the initial segment A_a is again an ordinal number. □

We will need the following proposition about well-ordered sets in the proof of Theorem 3.7

3.6 Proposition. *Let $A = (A, \leqslant_A)$ be an ordered set, and let $B = A \cup \{x\}$ be the one point extension of the set A (Definition 2.5).*

If the pair $(A, \leqslant)$ is well-ordered, then the pair $(B, \leqslant)$ is well-ordered, too.



Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

3.7 Theorem. *Let $A = (A, \leqslant_A)$ be an ordinal number, and let $A^+ := A \cup \{A\}$ be the successor set of the set A (Definition 2.11). Define the pair $(A^+, \leqslant_{A^+})$ to be the one point extension (Definition 2.5) of the pair $(A, \leqslant_A)$.*

Then the pair $(A^+, \leqslant_{A^+})$ is also an ordinal number.

Proof. By Proposition 3.6, the pair $(A^+, \leqslant_{A^+})$ is a well-ordered set.

Let a be an element of the set $A^+ = A \cup \{A\}$.

Case 1. *Suppose that the element a is contained in the set A.*

Since the pair $(A, \leqslant_A)$ is an ordinal number, it follows that

$$a = A_a = \{x \in A \mid x <_A a\} = \{x \in A^+ \mid x <_{A^+} a\} = (A^+)_a.$$

Case 2. *Suppose that $a = A$.*

Then we have

$$(A^+)_a = \{x \in A^+ \mid x <_{A^+} a\} = \{x \in A^+ \mid x <_{A^+} A\} = A = a.$$

Altogether, it follows that the pair $(A^+, \leqslant_{A^+})$ is an ordinal number. □

In Proposition 3.9 we will show that ordinal numbers are transitive set. Let us first recall what a transitive set is:

3.8 Definition. A set A is called a **transitive set** if every element of the set A is at the same time a subset of the set A.

French / German. Transitive set = Ensemble transitif = Transitive Menge.

3.9 Proposition. *Let $A = (A, \leqslant_A)$ be an ordinal number. If a is an element of the set A, then the element a is also a subset of the set A, that is, the set A is a transitive set (Definition 3.8).*

Proof. Let a be an element of the set A. Since the pair $(A, \leqslant)$ is an ordinal number, we have

$$a = A_a := \{x \in A \mid x < a\} \subseteq A.$$

□

Ordinal Numbers and Natural Numbers:

We will need the following theorem for the proof of Theorem 3.11:

3.10 Theorem. *The pair $(\mathbb{N}_0, \leqslant)$ is well-ordered (where $\leqslant$ denotes the standard order).*

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

3.11 Theorem. *The pair $(\mathbb{N}_0, \leqslant)$ with the standard order is an ordinal number.*

Proof. By Theorem 3.10, the set $\mathbb{N}_0$ is well-ordered. Let n be an element of the set $\mathbb{N}_0$. It follows from Theorem 2.13 that

$$(\mathbb{N}_0)_n = \{x \in \mathbb{N}_0 \mid x < n\} = n.$$

□

3.12 Theorem. *Let n be a natural number, and let $\leqslant$ be the standard order on the natural numbers. Then the pair $(n, \leqslant)$ (induced order) is an ordinal number.*

Proof. The assertion follows from Theorem 3.11 and from Proposition 3.5. □

3.13 Example. By Theorem 3.11, the set $\mathbb{N}_0$ is an ordinal number. It follows from Theorem 3.7 that the set $\mathbb{N}_0^+ := \mathbb{N}_0 \cup \{\mathbb{N}_0\}$ is also an ordinal number.

Historical Notes:

Ordinal numbers have been introduced by Georg Cantor. In a first step Cantor attributes to each totally ordered set an *order type* such that each totally ordered set has exactly one order type:

Jeder geordneten Menge M kommt ein bestimmter **Ordnungstypus** *oder kürzer ein bestimmter* **Typus** *zu, den wir mit*

$$\bar{M}$$

bezeichnen wollen; hierunter verstehen wir den Allgemeinbegriff, welcher sich aus M *ergibt, wenn wir nur von der Beschaffenheit der Elemente* m *abstrahieren, die Rangordnung unter ihnen aber beibehalten.*

See [Cantor 1895, p. 497].

Every ordered set M *has a certain* **order type** *or, more briefly, a certain* **type** *which we want to denote by*

$$\bar{M}.$$

By this we mean the general concept which results from M *if we only abstract from the nature of the elements* m*, but retain the order among them.*

Later on, he uses the definition of a type to introduce the ordinal numbers:

Nach §7 hat jede einfach geordnete Menge M *einen bestimmten Ordnungstypus* $\bar{M}$*; es ist dies der Allgemeinbegriff, welcher sich aus* M *ergibt, wenn unter Festhaltung der Rangordnung ihrer Elemente von der Beschaffenheit der letzteren abstrahiert wird, so dass aus ihnen lauter Einsen werden, die in einem bestimmten Rangverhältnis zueinander stehen. [...]*

Den Ordnungstyp einer wohlgeordneten Menge F *nennen wir die ihr zukommende* **Ordnungszahl**.

See [Cantor 1897, p. 216].

According to §7 every totally ordered set M has a certain order type $\bar{M}$*. This is the general concept that arises from* M *if, while maintaining the order of its elements, the nature of the latter is abstracted, so that they become all ones that are in a certain order relation to one another. [...]*

We call the order type of a well-ordered set F *the* **ordinal number** *assigned to it.*

This definition is a bit vague since it is not really defined what the type of a well-ordered set is. Since isomorphism between ordered sets is reflexive, symmetric and transitive, one could think of the type of an ordered set as its equivalence class with respect to isomorphism. However, here we have the problem that neither the ordered sets nor even the well-ordered sets form a set (see Theorem 6.3) with the consequence that we cannot define an equivalence relation on the set of the well-ordered sets (since it does not exist) and therefore we do not get an equivalence class.

It was John von Neumann who found a solution by giving an alternative definition of ordinal numbers:

Das Ziel der vorliegenden Arbeit ist: den Begriff der Cantorschen Ordnungszahl eindeutig und konkret zu fassen. [...]

Wir wollen eigentlich den Satz "Jede Ordnungszahl ist der Typus der Menge aller ihr vorangehenden Ornungszahlen" zur Grundlage unserer Überlegungen machen. Damit aber der vage Begriff "Typus" vermieden werde, in dieser Form: "Jede Ordnungszahl ist die Menge der Menge aller ihr vorangehenden Ornungszahlen."

See [von Neumann 1922, p. 199].

The aim of the present work is to define Cantor's term ordinal number clearly and concretely. [...]

We actually want to make the sentence "Every ordinal number is the type of the set of all preceding ordinal numbers" as the basis of our considerations. But so that the vague *term "type" is avoided, in this form: "Every ordinal number is the set of the set of all ordinal numbers preceding it."*

von Neumann's formal definition of an ordinal number is as follows:

Ξ sei eine wohlgeordnete Menge. Wir nennen eine Funktion $f(x)$, die in Ξ definiert ist, eine **Zählung** *von Ξ, wenn für alle Elemente x von Ξ*

$$f(x) = M\big(f(y); y \in A(x, \Xi)\big)$$

ist. Wenn $f(x)$ eine Zählung von Ξ ist, so nennen wir

$$M\big(f(x); x \in \Xi\big)$$

eine **Ordnungszahl** *von Ξ.*

See [von Neumann 1922, p. 200].

Let Ξ be a well-ordered set. We call a function $f(x)$, defined in Ξ, a **count** *of Ξ if for all elements x of Ξ we have*

$$f(x) = M\big(f(y); y \in A(x, \Xi)\big).$$

If $f(x)$ is a count of Ξ, then we call

$$M\big(f(x); x \in \Xi\big)$$

an **ordinal number** *of Ξ.*

Let us translate von Neumann's terminology into our terminology: $A(x, \Xi)$ stands for the initial segment Ξ_x of the set Ξ with respect to the element x. $M\big(f(x); x \in \Xi\big)$ simply means $\{f(x) \mid x \in \Xi\}$. Hence, we may formulate the definition of von Neumann as follows:

Let A be a well-ordered set, and let $f : A \to B$ be a function with the property that

$$f(x) = \{f(y) \mid y \in A_x\},$$

then the function $f : A \to B$ is called a **count** of the set A. In this case the set $\{f(x) \mid x \in A\}$ is called an ordinal number.

von Neumann does not explain what the set B is, but he refers to the axiom of substitution (see Unit *Families and the Axiom of Choice* [Garden 2020c]) for its existence.

Finally, von Neumann obtains the following characterization of ordinal numbers:

9. P *ist dann und nur dann eine Ordnungszahl, wenn es*

1. eine durch Subsumption ordnungsfähige Menge von Mengen ist,

2. seine Subsumptionsordnung eine Wohlordnung ist,

3. für jedes Element ξ von P stets $\xi = A(\xi, P)$ ist.

See [von Neumann 1922, p. 203].

9. P *is an ordinal number if and only if*

1. it is a set of sets ordered by subsumption,

2. its subsumption order is a well-ordering,

3. for every element ξ of P there is always $\xi = A(\xi, P)$.

A set P of sets is called *ordered by subsumption* if we have $A \subseteq B$ or $B \subseteq A$ for all elements (sets) of the set P. The subsumption order of P is the order $\subseteq$. This characterization is almost the same as our Definition 3.2 of an ordinal number.

4 Isomorphisms between Ordinal Numbers

We will explain the following two important results about ordinal numbers:

(i) Isomorphic ordinal numbers are equal (Theorem 4.2).

(ii) For two diferent ordinal numbers A and B we either have $A \in B$ and $A \subset B$ or we have $B \in A$ and $B \subset A$ (Theorem 4.6). In particular, we have for each two ordinal numbers that

$$A \in B \Leftrightarrow A \subset B.$$

See Theorem 4.9.

Isomorphisms between Ordinal Numbers:

We shall prove Theorem 4.2 by transfinite induction. Let us first recall this method:

4.1 Theorem. (Principle of Transfinite Induction) *Let $(A, \leqslant)$ be a well-ordered set. For an element z of the set A, we denote by*

$$A_z := \{x \in A \mid x < z\}$$

the initial segment of the set A with respect to the element z. Let B be a subset of the set A fulfilling the following condition:

$A_z \mapsto z$: For all elements z of the set A, it follows from $A_z \subseteq B$ that the element z is contained in the set B.

Then we have $B = A$.

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

4.2 Theorem. *Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordinal numbers which are isomorphic as ordered sets (Definition 2.6).*

(a) We have $(A, \leqslant_A) = (B, \leqslant_B)$, that is, the ordinal numbers are identical.

(b) Let $\alpha : (A, \leqslant_A) \to (B, \leqslant_B)$ be an isomorphism from the ordered set A onto the ordered set B. Then we have $\alpha(x) = x$ for all elements x of the set A.

Proof. We shall prove (a) and (b) in common by transfinite induction. For, let

$$\alpha : (A, \leqslant_A) \to (B, \leqslant_B)$$

be an isomorphism from the ordered set A onto the ordered set B, and let

$$M := \{x \in A \mid \alpha(x) = x\}.$$

Step 1. *We will show by transfinite induction that $M = A$:*

$A_a \mapsto a$: Let a be an element of the set A such that the initial segment $A_a := \{x \in A \mid x < a\}$ is a subset of the set M. Then we have

$$\alpha(x) = x \text{ for all } x \in A_a.$$

Let $b := \alpha(a)$. Since the set B is an ordinal number, we have $b = B_b$. We claim that $a = A_a = B_b = b$:

In order to show that the initial segment A_a is a subset of the initial segment B_b, let x be an element of the initial segment A_a. Since $x < a$ and $\alpha(x) = x$, it follows that

$$x = \alpha(x) < \alpha(a) = b.$$

Hence, the element x is contained in the initial segment B_b.

In order to show that the initial segment B_b is a subset of the initial segment A_a, let y be an element of the initial segment B_b, and let $x := \alpha^{-1}(y)$. Since $y < b$, we have

$$x = \alpha^{-1}(y) < \alpha^{-1}(b) = a,$$

that is, the element x is contained in the initial segment A_a. It follows that

$$y = \alpha(x) = x \in A_a.$$

Hence, the element y is contained in the initial segment A_a.

It follows that

$$\alpha(a) = b = B_b = A_a = a.$$

Hence, the element a is contained in the set M. By transfinite induction, we get $M = A$.

Step 2. *We have* $(A, \leqslant_A) = (B, \leqslant_B)$*:*

It follows from Step 1 that

$$B = \{\alpha(x) \mid x \in A\} = \{x \mid x \in A\} = A.$$

The equation $(A, \leqslant_A) = (B, \leqslant_B)$ follows from the fact that the mapping $\alpha : A \to B$ is an isomorphism. □

4.3 Remark. My first idea for a short proof of Theorem 4.2 was the equation

$$\alpha(a) = \alpha(A_a) = \{\alpha(x) \mid x \in A_a\} = \{x \mid x \in A_a\} = A_a = a.$$

But, the equation $\alpha(A_a) = \{\alpha(x) \mid x \in A_a\}$ is not correct. In this context the set A_a is an element of the set A which is mapped by the mapping α to an element of the set B. The equation $\alpha(A_a) = \{\alpha(x) \mid x \in A_a\}$ is a second meaning of the term $\alpha(A_a)$ which may not be applied here.

For more details see Unit *Functions and Equivalent Sets* [Garden 2020b].

4.4 Proposition. *Let* $(A, \leqslant)$ *be an ordinal number. Then the automorphism group* $\mathrm{Aut}(A)$ *of* A *is trivial, that is,* $\mathrm{Aut}(A) = \{\mathrm{id}\}$.

Proof. The assertion follows from Theorem 4.2. □

Relation between two arbitrary Ordinal Numbers:

We will need the following result about well-ordered sets in the proof of Theorem 4.6:

4.5 Theorem. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two well-ordered sets. Then one of the following cases occurs:*

(i) The sets A *and* B *are isomorphic.*

(ii) There exists an element a *of the set* A *such that the set* B *is isomorphic to the initial segment* $A_a := \{x \in A \mid x <_A a\}$ *of the set* A.

(iii) There exists an element b *of the set* B *such that the set* A *is isomorphic to the initial segment* $B_b := \{y \in B \mid y <_B b\}$ *of the set* B.

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

4.6 Theorem. *Let* $A = (A, \leqslant_A)$ *and* $B = (B, \leqslant_B)$ *be two ordinal numbers. Then exactly one of the following cases occurs:*

(i) We have $(A, \leqslant_A) = (B, \leqslant_B)$.

(ii) We have $A = b = B_b$ *for some element* b *of the set* B*, and the order* $\leqslant_A$ *is induced by the order* $\leqslant_B$*. In particular, we have* $A \in B$ *and* $A \subset B$.

(iii) We have $B = a = A_a$ *for some element* a *of the set* A*, and the order* $\leqslant_B$ *is induced by the order* $\leqslant_A$*. In particular, we have* $B \in A$ *and* $B \subset A$.

Proof. Since the pairs $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are well-ordered set, it follows from Theorem 4.5 that one of the following cases occurs:

(i) We have $(A, \leqslant_A) \cong (B, \leqslant_B)$.

(ii) The pair $(A, \leqslant_A)$ is isomorphic to the pair $(B_b, \leqslant_B)$ where B_b denotes the initial segment with respect to some element b of the set B and $\leqslant_B$ is the order on the initial segment B_b induced by the order $\leqslant_B$ on the set B.

(iii) The pair $(B, \leqslant_B)$ is isomorphic to the pair $(A_a, \leqslant_A)$ where A_a denotes the initial segment with respect to some element a of the set A and $\leqslant_A$ is the order on the initial segment A_a induced by the order $\leqslant_A$ on the set A.

Case (i): It follows from Theorem 4.2 that $(A, \leqslant_A) = (B, \leqslant_B)$.

Case (ii): By Proposition 3.4, the pair $(B_b, \leqslant_B)$ (induced order) is an ordinal number. Since B is an ordinal number, we have $b = B_b$. By Theorem 4.2, it follows from $(A, \leqslant_A) \cong (B_b, \leqslant_B)$ that $(A, \leqslant_A) = (B_b, \leqslant_B)$.

Case (iii) is analogous to Case (ii). □

4.7 Remark. For two *ordinal numbers* $(A, \leqslant_A)$ and $(B, \leqslant_B)$, it follows from Theorem 4.6 that we do not have to distinguish between the orders $\leqslant_A$ and $\leqslant_B$ because we either have $\leqslant_A = \leqslant_B$ or we have $A \subset B$ and the order $\leqslant_A$ is induced by the order $\leqslant_B$ or we have $B \subset A$ and the order $\leqslant_B$ is induced by the order $\leqslant_A$.

We therefore will denote an ordinal number $(A, \leqslant_A)$ just by $(A, \leqslant)$. Note that this cannot be extended to *well-ordered sets* in general.

In the proof of Theorem 4.9 we will need the following consequence of the axiom of foundation:

4.8 Theorem. *Let* A *and* B *be two sets.*

(a) IF $A \in B$*, then we have* $A \neq B$.

(b) If $A \in B$*, then we have* $B \notin A$.

Proof. See Unit *Unions and Intersections of Sets* [Garden 2020a]. ☐

4.9 Theorem. *Let* $\mathcal{A} = (A, \leqslant)$ *and* $\mathcal{B} = (B, \leqslant)$ *be two ordinal numbers. Then the following two conditions are equivalent:*

(i) The set A *is an element of the set* B*, that is,* $A \in B$.

(ii) The set A *is a proper subset of the set* B*, that is,* $A \subset B$.

Proof. (i) $\Rightarrow$ (ii) Suppose that $A \in B$. By Theorem 4.8, we have $A \neq B$ and $B \notin A$. It follows from Theorem 4.6, that we have $A \in B$ and $A \subset B$.

(ii) $\Rightarrow$ (i) Suppose that $A \subset B$. It follows that $A \neq B$ and $B \not\subset A$. By Theorem 4.6, we have $A \in B$ and $A \subset B$. ☐

Historical Notes:

Theorem 4.2 (isomorphic ordinal numbers are equal) is due to von Neumann:

14. Ξ *und* H *seien wohlgeordnete Mengen.* Ξ *und* H *sind dann und nur dann einander ähnlich, wenn* $OZ(\Xi) = OZ(H)$ *ist.*

See [von Neumann 1922, p. 206].

14. *Let* Ξ *and* H *be two well-ordered sets.* Ξ *and* H *are similar if and only if* $OZ(\Xi) = OZ(H)$.

Similar means isomorphic. $OZ(X)$ denotes the ordinal number isomorphic to the well-ordered set X. von Neumann has also shown that ordinal numbers are well-ordered. So Theorem 4.2 is an easy consequence of Theorem 14 of von Neumann.

Theorem 4.6 (two ordinal numbers are either equal or one is an initial segment of the other) is due to Cantor:

A. *Sind* α *und* β *zwei beliebige Ordnungszahlen, so ist entweder* $\alpha = \beta$ *oder* $\alpha < \beta$ *oder* $\beta < \alpha$.

See [Cantor 1897, p. 216].

A. *If* α *und* β *are two arbitrary ordinal numbers, then we have either* $\alpha = \beta$ *or* $\alpha < \beta$ *or* $\beta < \alpha$.

Note that in this paper of Cantor $\alpha < \beta$ means that the ordinal number α is an initial segment of the ordinal number β.

Theorem 4.6 has also been shown by von Neumann:

15. Ξ *ist dann und nur dann einem Abschnitt in* H *ähnlich, wenn* $OZ(\Xi) \subset OZ(H)$ *ist.*

See [von Neumann 1922, p. 206].

15. Ξ *is similar to an initial segment of* H *if and only if* $OZ(\Xi) \subset OZ(H)$.

The sets Ξ and H are assumed to be well-ordered. As before Theorem 4.6 is an easy consequence of Theorem 15 of von Neumann.

Theorem 4.9 (an element of an ordinal number is a proper subset of this ordinal number and vice versa) is due to von Neumann:

10. *[...] Wir können den jetzt bewiesenen Satz auch so aussprechen: Wenn* P, Q *Ordnungszahlen sind, so ist* $P \subset Q$ *mit* $P \in Q$ *gleichbedeutend.*

See [von Neumann 1922, p. 204].

10. *We can formulate the theorem that has just been proven as follows: If* P *and* Q *are ordinal numbers, then* $P \subset Q$ *is equivalent to* $P \in Q$*.*

5 The Standard Order on the Ordinal Numbers

We will define the standard order on the ordinal numbers (Definition 5.1), and we will see that the first ordinal numbers are the natural numbers followed by the set $\mathbb{N}_0$ of all natural numbers (Theorem 5.5). The most important result of this section is the fact that every set of ordinal numbers is well-ordered (Theorem 5.6).

Definition of the Standard Order on the Ordinal Numbers:

5.1 Definition. Let $A = (A, \leqslant)$ and $B = (B, \leqslant)$ be two ordinal numbers. We define the order

$$A \leqslant B \text{ if and only if } A \subseteq B.$$

The so defined order $\leqslant$ is called the **standard order on the ordinal numbers** (see Remark 5.4).

French / German. Standard order on the ordinal numbers = Ordre standard sur les nombres ordinals = Standardordnung auf den Ordinalzahlen.

Elementary Properties of the Order on the Ordinal Numbers:

5.2 Proposition. *Let* $(A, \leqslant)$, $(B, \leqslant)$ *and* $(C, \leqslant)$ *be three ordinal numbers.*

(a) We have $A \leqslant A$.

(b) If $A \leqslant B$ *and* $B \leqslant A$*, then we have* $A = B$.

(c) If $A \leqslant B$ *and* $B \leqslant C$*, then we have* $A \leqslant C$.

Proof. The assertions follow directly from the definition that $A \leqslant B$ if and only if $A \subseteq B$. □

5.3 Theorem. *Let* $\mathcal{A}$ *be a set of ordinal numbers. Then the standard order on the ordinal numbers (Definition 5.1) is a total order on the set* $\mathcal{A}$.

Proof. By Proposition 5.2, the relation $\leqslant$ is an order on the set $\mathcal{A}$. It follows from Theorem 4.6 that we have

$$A \subseteq B \text{ or } B \subseteq A$$

for all ordinal numbers A and B of the set $\mathcal{A}$ implying that we have $A \leqslant B$ or $B \leqslant A$ for all ordinal numbers A and B of the set $\mathcal{A}$. □

5.4 Remark. We shall see in Theorem 6.3 that there does not exist a set containing all ordinal numbers. Hence, the standard order on the ordinal numbers (Definition 5.1) is not an order on the set of all ordinal numbers (because this set does not exist), but an order on every set of ordinal numbers (Theorem 5.3).

5.5 Theorem. *Let A be an ordinal number. Then exactly one of the following possibilities occurs:*

(i) There is a natural number n such that $A = n$.

(ii) We have $\mathbb{N}_0 \leqslant A$.

Proof. By Theorem 5.3, the set $\{A, \mathbb{N}_0\}$ is totally ordered. In particular, we have $A < \mathbb{N}_0$ or $\mathbb{N}_0 \leqslant A$. If $A < \mathbb{N}_0$, then, by Theorem 4.6, there exists an element n of the set $\mathbb{N}_0$ such that

$$A = (\mathbb{N}_0)_n = \{x \in \mathbb{N}_0 \mid x < n\}.$$

By Theorem 2.13, we have

$$A = \{x \in \mathbb{N}_0 \mid x < n\} = n.$$

□

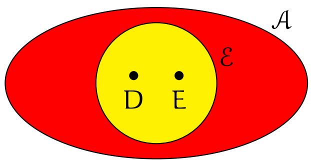
5.6 Theorem. *Let $\mathcal{A}$ be a set of ordinal numbers. Then the pair $(\mathcal{A}, \leqslant)$ is well-ordered where $\leqslant$ denotes the standard order on the ordinal numbers.*

Proof. Let $\mathcal{E}$ be a non-empty subset of the set $\mathcal{A}$, and let E be an element of the set $\mathcal{E}$.

Case 1. *Suppose that we have $E \leqslant X$ for all elements X of the set $\mathcal{E}$.*

Then the set E is a minimal element of the set $\mathcal{E}$, and there is nothing to show.

Case 2. *Suppose that there exists an element D of the set $\mathcal{E}$ such that $D < E$.*



By Theorem 4.9, the element D is contained in the set E implying that the set $E \cap \mathcal{E}$ is non-empty. Since the set E is well-ordered, there exists a minimal element M of the set $E \cap \mathcal{E}$. In particular, M is an element of the set $\mathcal{E}$ implying that the set M is an ordinal number.

Since the set M is a minimal element of the set $E \cap \mathcal{E}$ and since the set D is contained in the set $E \cap \mathcal{E}$, it follows that $M \leqslant D$. Since $D < E$, we get $M < E$.

Let X be an element of the set $\mathcal{E}$. If $E \leqslant X$, then we have $M < E \leqslant X$.

If $X < E$, then, by Theorem 4.9, the element X is contained in the set E. In particular, the set X is contained in the set $E \cap \mathcal{E}$ implying that $M \leqslant X$.

Hence, the element M is the minimal element of the set $\mathcal{E}$ implying that the set $\mathcal{A}$ is well-ordered. □

Historical Notes:

Theorem 5.6 (a set of ordinal numbers is well-ordered) is due to von Neumann:

12. *Sei* U *eine Menge von Ordnungszahlen. [...] Die Subsuptionsordnung von* U *ist aber eine Wohlordnung.*

See [von Neumann 1922, p. 205].

12. *Let* U *be a set of ordinal numbers. [...] The ordering by subsumption is a well-ordering.*

Note that the (quite tricky) proof of Theorem 5.6 also stems from von Neumann.

6 The Theorem of Burali-Forti

The two main results of this section are the fact that the union of a family of ordinal numbers is again an ordinal number (Theorem 6.1) and the Theorem of Burali-Forti saying that the ordinal numbers do not form a set (Theorem 6.3).

The Union of Ordinal Numbers:

The following theorem is an important tool to construct new ordinal numbers from given ones:

6.1 Theorem. *Let* $\mathcal{A}$ *be a set of ordinal numbers, and let*

$$S := \bigcup_{X \in \mathcal{A}} X.$$

(a) The set S *is an ordinal number.*

(b) The ordinal number S *is the only ordinal number with the following properties:*

(i) We have $A \leqslant S$ *for all ordinal numbers* A *of the set* $\mathcal{A}$.

(ii) If T *is an ordinal number such that* $A \leqslant T$ *for all ordinal numbers* A *of the set* $\mathcal{A}$, *then we have* $S \leqslant T$

Note that the ordinal number S *is not necessarily contained in the set* $\mathcal{A}$.

Proof. (a) By Proposition 3.5, every element of an ordinal number is an ordinal number. It follows that each element $\emptyset \neq X$ of the set $\mathcal{A}$ is a set of ordinal numbers implying that the set

$$S := \bigcup_{X \in \mathcal{A}} X$$

is a set of ordinal numbers.

It follows from Theorem 5.6 that the set S is well-ordered. In order to show that the set S is an ordinal number, let s be an element of the set S. We have to show that

$$s = S_s \text{ where } S_s := \{x \in S \mid x < s\}.$$

Since we have

$$s \in S = \bigcup_{X \in \mathcal{A}} X,$$

there exists a set A of the set $\mathcal{A}$ containing the element s. Since the set A is an ordinal number, we have $s = A_s := \{x \in A \mid x < s\}$. Since the set A is a subset of the set S, we have

$$A_s = \{x \in A \mid x < s\} \subseteq \{x \in S \mid x < s\} = S_s.$$

In order to show that the set S_s is a subset of the set A_s, let t be an element of the set S_s. Since

$$t \in S = \bigcup_{X \in \mathcal{A}} X,$$

there exists a set B of the set $\mathcal{A}$ containing the element t.

If the set B is a subset of the set A, then the element t is contained in the set A.

If the set A is a subset of the set B, then the element s is contained in the set B. Since the set B is an ordinal number, we have

$$s = B_s := \{x \in B \mid x < s\}.$$

Since the element t is contained in the set S_s, we have $t < s$ implying that

$$t \in B_s = s = A_s.$$

It follows that the element t is contained in the set A_s implying that the set S_s is a subset of the set A_s. Since the set A_s is a subset of the set S_s, we obtain

$$s = A_s = S_s.$$

(b) Let

$$S := \bigcup_{X \in \mathcal{A}} X.$$

By (a), the set S is an ordinal number.

Step 1. *We have $A \leqslant S$ for all sets A of the set $\mathcal{A}$:*

Let A be a set of the set $\mathcal{A}$. Since we have

$$X \leqslant S \text{ if and only if } X \subseteq S \text{ for all ordinal numbers } X \text{ (Definition 5.1)},$$

it follows from

$$A \subseteq \bigcup_{X \in \mathcal{A}} X = S$$

that the set A is a subset of the set S. In particular, we have $A \leqslant S$.

Step 2. *Let T be an ordinal number such that*

$$X \leqslant T \textit{ for all } X \in \mathcal{A}.$$

Then we have $S \leqslant T$:

It follows from $X \leqslant T$ for all sets X of the set $\mathcal{A}$ that

$$S = \bigcup_{X \in \mathcal{A}} X \subseteq T,$$

hence, $S \subseteq T$ implying that $S \leqslant T$.

Step 3. *The ordinal number S is uniquely determined:*

Obviously, the element S is the maximum (Definition 2.3) of the set $\mathcal{A} \cup \{S\}$. It follows from Proposition 2.4 that the ordinal number S is uniquely determined. □

6.2 Examples. (a) Let $\mathcal{A} := \{0, 3, 5\}$. The set $\mathcal{A}$ is a set of ordinal numbers, but no ordinal number itself. We have

$$\bigcup_{X \in \mathcal{A}} X = 0 \cup 3 \cup 5 = \emptyset \cup \{0, 1, 2\} \cup \{0, 1, 2, 3, 4\} = \{0, 1, 2, 3, 4\} = 5$$

which is of course an ordinal number.

(b) Let $\mathcal{B} := \{2(n+1) \mid n \in \mathbb{N}_0\}$. The set $\mathcal{B}$ is a set of ordinal numbers, but no ordinal number itself. We have

$$\bigcup_{X \in \mathcal{B}} X = \bigcup_{n \in \mathbb{N}_0} 2(n+1) = \bigcup_{n \in \mathbb{N}_0} \{0, 1, \dots, 2n+1\} = \mathbb{N}_0$$

which is of course an ordinal number.

The Non-Existence of the Set of all Ordinal Numbers:

6.3 Theorem. (Burali-Forti) *The ordinal numbers do not form a set. In other words, there is not set $\mathcal{A}$ with the property that*

$$A \in \mathcal{A} \text{ if and only if } A \text{ is an ordinal number.}$$

Proof. Assume that there exists a set $\mathcal{A}$ of all ordinal numbers. By Theorem 6.1, the set

$$S := \bigcup_{X \in \mathcal{A}} X.$$

is an ordinal number with

$$X \leqslant S \text{ for all } X \in \mathcal{A}.$$

Let S^+ be the successor of the set S. By Theorem 3.7, the set S^+ is an ordinal number such that $S < S^+$. Since the set $\mathcal{A}$ consists of all ordinal numbers, this yields a contradiction. □

Historical Notes:

Theorem 6.3 (the ordinal numbers do not form a set) is due to Cesare Burali-Forti. The information is distributed over the paper [Burali-Forti 1897]:

[...] allora $(\mathrm{No}, \bar{\varepsilon} >)$ *indica la classe dei numeeri ordinali ordinati in senso crescente.*

See [Burali-Forti 1897, p. 163].

[...] then $(\mathrm{No}, \bar{\varepsilon} >)$ *indicates the class of ordinal numbers in ascending order.*

In other words, Burali-Forti assumes that the set $(\mathrm{No}, \bar{\varepsilon} >)$ of the ordinal numbers exists. He has already shown that we have $X \subseteq Y$ or $Y \subseteq X$ for all ordinal numbers X and Y.

32. $(\mathrm{No}, \bar{\varepsilon} >) \in \mathrm{Kpo}$.

See [Burali-Forti 1897, p. 163].

Kpo stands for the class of well-ordered sets. $(\mathrm{No}, \bar{\varepsilon} >) \in \mathrm{Kpo}$ means that the set $(\mathrm{No}, \bar{\varepsilon} >)$ is well-ordered.

[...] possiamo porre

33. $\Omega = \mathrm{T}'(\mathrm{No}, \bar{\varepsilon} >)$.

See [Burali-Forti 1897, p. 163].

[...] we can set

33. $\Omega = T'(\mathrm{No}, \bar{\varepsilon} >)$.

$T'(X)$ stands for the ordinal number of a well-ordered set. This means that Ω denotes the ordinal number of the set $(\mathrm{No}, \bar{\varepsilon} >)$ of all ordinal numbers in ascending order.

[...] abbiamo, in virtù delle prop. 34, 26, 29

$$\Omega + 1 > \Omega; \; \Omega + 1 \leqslant \Omega$$

che, per le prop. 21, 22, risultano contraddittorie.

See [Burali-Forti 1897, p. 164].

[...] we have, in view of Proposition 34, 26, 29

$$\Omega + 1 > \Omega; \; \Omega + 1 \leqslant \Omega$$

which yields a contradiction in view of Proposition 21, 22.

This contradiction shows that there is no set of all ordinal numbers. Burali-Forti summarizes his result as follows:

[...] i tipi d'ordine non possono fornire per le classi ordinate una classe campione, come per le classi finite e numerabili le fornisce la classe dei numeri interni ordinata in senso crescente.

See [Burali-Forti 1897, p. 164].

[...] the ordinal numbers cannot provide a sample class for the ordered sets, as for the finite and countable sets the ascending class of internal numbers supplies it.

7 Well-Ordered Sets and Ordinal Numbers

The main result of this section is Theorem 7.7 saying that every well ordered set is isomorphic to exactly one ordinal number. The other results in this section prepare the proof of this theorem.

7.1 Theorem. (Well-Ordering Theorem) *Every set A can be well-ordered, that is, there exists an order $\leqslant_A$ on the set A such that the pair $(A, \leqslant_A)$ is well-ordered.*

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

7.2 Proposition. *Let A be an ordered set, and let $B := A \cup \{b\}$ be the one point extension of the set A where b is an element of the set B not contained in the set A. Suppose that the set A is isomorphic to an ordinal number D.*

Then the set B is isomorphic to the ordinal number $E = D^+$ (successor of the set D, see Definition 2.11).

Proof. By Theorem 3.7, the successor $E := D^+ := D \cup \{D\}$ of the set D is an ordinal number where the order $\leqslant_E$ on the set E is defined as follows: For each two elements x and y of the set D, set $x \leqslant_E y$ if and only if $x \leqslant_D y$. For each element x of the set E, set $x \leqslant_E D$.

Since the sets A and D are isomorphic, there exists an isomorphism $\alpha : A \to D$ from the ordered set A onto the ordered set D. It follows that the function $\beta : B \to E$ defined by

$$\beta(x) := \begin{cases} \alpha(x) & \text{if } x \in A \\ D & \text{if } x = b \end{cases}$$

is an isomorphism from the ordered set B onto the ordered set E.

Hence, the set B is isomorphic to the ordinal number $E = D^+$. □

7.3 Proposition. *Let A be a well-ordered set, and let s and t be two elements of the set A such that $s < t$. Suppose that there exist two ordinal numbers B and C and two isomorphisms*

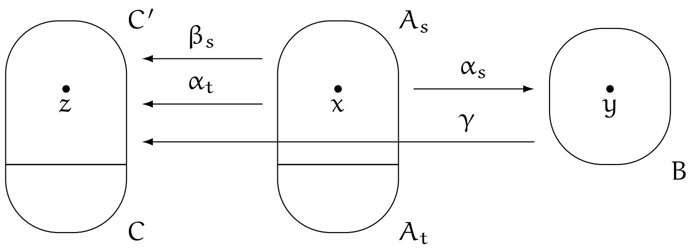
$$\alpha_s : A_s \to B \text{ and } \alpha_t : A_t \to C$$

from the initial segments A_s and A_t onto the ordinal numbers B and C, respectively.

Then we have $\alpha_t(A_s) = B$ and

$$\alpha_t(x) = \alpha_s(x) \text{ for all } x \in A_s.$$

Proof. Let $\beta_s := \alpha_t|_{A_s} : A_s \to C' := \alpha_t(A_s)$ be the restriction of the function $\alpha_t : A_t \to C$ to the set A_s, and let $\gamma : B \to C'$ be defined by $\gamma := \beta_s \circ \alpha_s^{-1}$.



Since the functions $\alpha_s : A_s \to B$ and $\beta_s : A_s \to C'$ are isomorphisms from the ordered set A_s onto the ordered sets B and C', respectively, the function $\gamma : B \to C'$ is an isomorphism from the ordered set B onto the ordered set C.

By Proposition 2.10, the set C' is an initial segment of the set C. Since the set C is an ordinal number, it follows from Proposition 3.4 that the set C' is also an ordinal number.

By Theorem 4.2, we have $B = C'$ and $\gamma = \mathrm{id}$. Let x be an element of the set A_s, and let $y := \alpha_s(x)$ and $z := \beta_s(x)$. It follows that

$$y = \gamma(y) = \left(\beta_s \circ \alpha_s^{-1}\right)(y) = \beta_s\left(\alpha_s^{-1}(y)\right) = \beta_s(x) = z.$$

Hence, we get

$$\alpha_s(x) = \beta_s(x) = \alpha_t(x) \text{ for all } x \in A_s.$$

□

Proposition 7.6 prepares the proof of Theorem 7.7 saying that every well-ordered set is isomorphic to an ordinal number. We will need Proposition 7.4 and Theorem 7.5 about well-ordered sets for the proof of Proposition 7.6.

7.4 Proposition. *Let* I *be a totally ordered index set, and let* $\big((A_i, \leqslant_{A_i})\big)_{i\in I}$ *and* $\big((B_i, \leqslant_{B_i})\big)_{i\in I}$ *be two families of ordered sets with the following properties:*

(i) If i *and* j *are two elements of the set* I *such that* $i \leqslant j$*, then the set* A_i *is a subset of the set* A_j*, the order of the set* A_i *is induced by the order of the set* A_j*, the set* B_i *is a subset of the set* B_j*, and the order of the set* B_i *is induced by the order of the set* B_j*.*

(ii) For each element i *of the set* I*, there exists an isomorphism* $\alpha_i : A_i \to B_i$*.*

(iii) For each two elements i *and* j *of the set* I *such that* $i \leqslant j$*, we have* $\alpha_j|_{A_i} = \alpha_i$*, that is, we have*

$$\alpha_j(x) = \alpha_i(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i\in I} A_i \text{ and } B := \bigcup_{i\in I} B_i,$$

and let $\leqslant_A$ *and* $\leqslant_B$ *be the orders on the sets* A *and* B *induced by the chains of ordered sets* $(A_i)_{i\in I}$ *and* $(B_i)_{i\in I}$*, respectively.*

Then there exists exactly one isomorphism

$$\alpha : (A, \leqslant_A) \to (B, \leqslant_B)$$

from the ordered set A *onto the ordered set* B *such that* $\alpha_i = \alpha|_{A_i}$*, that is,*

$$\alpha(x) = \alpha_i(x) \text{ for all } x \in A_i \text{ for all } i \in I.$$

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

7.5 Theorem. *Let* $A = (A, \leqslant)$ *be a well-ordered set. Then one of the following two cases occurs:*

(i) The set A *has a maximal element* z*, and we have*

$$A = A_z \cup \{z\}$$

where $A_z := \{x \in A \mid x < z\}$ *denotes the initial segment of the set* A *with respect to the element* z*.*

(ii) There does not exist a maximal element of the set A*, and we have*

$$A = \bigcup_{x\in A} A_x.$$

Proof. See Unit *Well-Ordered Sets* [Garden 2020h]. □

7.6 Proposition. *Let* A *be a well-ordered set, and suppose that the initial segment* A_x *is isomorphic to an ordinal number* B_x *for all elements* x *of the set* A*.*

Then the set A *is isomorphic to an ordinal number.*

Proof. Since the set A is well-ordered, it follows from Theorem 7.5 that one of the following cases occurs:

(i) There exists an element z of the set A such that $A = A_z \cup \{z\}$.

(ii) We have $A = \bigcup_{z \in A} A_z$.

Case (i): By assumption, the ordered set A_z is isomorphic to an ordinal number B_z. It follows from Proposition 7.2 that the set A is isomorphic to the ordinal number $B := (B_z)^+$.

Case (ii): Obviously, we have

$$A_r = \{x \in A \mid x < r\} \subseteq A_s = \{x \in A \mid x < s\} \text{ whenever } r \leqslant s.$$

It follows from Proposition 7.3 that

$$B_r \subseteq B_s \text{ if } r \leqslant s.$$

For each element z of the set A, let $\alpha_z : A_z \to B_z$ be an isomorphism from the ordered set A_z onto the ordinal number B_z. Let r and s be two elements of the set A such that $r < s$. It follows from Proposition 7.3 that we have

$$\alpha_r(x) = \alpha_s(x) \text{ for all } x \in A_r.$$

It follows from Proposition 7.4 that there exists an isomorphism

$$\alpha : A := \bigcup_{z \in A} A_z \to B := \bigcup_{z \in A} B_z.$$

from the ordered set A onto the ordered set B. By Theorem 6.1, the set B is an ordinal number. Hence, the set A is isomorphic to the ordinal number B. □

7.7 Theorem. *Let $A = (A, \leqslant_A)$ be a well-ordered set. Then there exists exactly one ordinal number $B = (B, \leqslant)$ isomorphic to the set A.*

Proof. Let A be a well-ordered set. As usually, we denote by $A_z := \{x \in A \mid x < z\}$ the initial segment of the set A with respect to an element z of the set A.

Step 1. *There exists at most one ordinal number B isomorphic to the set A:*

Let B_1 and B_2 be two ordinal numbers such that $B_1 \cong A$ and $B_2 \cong A$. It follows from Proposition 2.7 that $B_1 \cong B_2$. By Theorem 4.2, it follows that $B_1 = B_2$.

Step 2. *For each element z of the set A, the initial segment A_z is isomorphic to an ordinal number:*

Let

$$Z := \{z \in A \mid A_z \text{ is isomorphic to an ordinal number}\}.$$

We have to show that $Z = A$. Since the set A is well-ordered, we may proceed by transfinite induction:

$A_z \mapsto z$: Let z be an element of the set A, and suppose that the initial segment A_z is contained in the set Z. We have to show that the element z is contained in the set Z, that is, that the initial segment A_z is isomorphic to an ordinal number:

Since the set A_z is a subset of the set Z, it follows that for each element y of the set A_z the initial segment $(A_z)_y = A_y$ (Proposition 2.9) is isomorphic to an ordinal number.

By Proposition 7.6, the set ordered set A_z is isomorphic to an ordinal number.

Step 3. *The set A is isomorphic to an ordinal number:*

By Step 2, each initial segment A_z of the set A is isomorphic to an ordinal number. It follows from Proposition 7.6 that the ordered set A is isomorphic to an ordinal number. □

7.8 Remark. By the well-ordering theorem (Theorem 7.1), every set A can be endowed with a well-ordering $\leqslant$ such that the pair $(A, \leqslant)$ is a well ordered set. In view of Theorem 7.7, the pair $(A, \leqslant)$ is isomorphic to exactly one ordinal number N. Hence, we can associate to each set A an ordinal number N. However, this procedure is not unique:

A set A can be endowed with different well-orderings $\leqslant_1$ and $\leqslant_2$ such that the pairs $(A, \leqslant_1)$ and $(A, \leqslant_2)$ are not isomorphic. As a consequence, the pairs $(A, \leqslant_1)$ and $(A, \leqslant_2)$ are isomorphic to different ordinal numbers.

For example, let $\leqslant_1$ be the standard order on the set $\mathbb{N}_0$, and let $\leqslant_2$ be defined as follows:

$$m \leqslant_2 n \text{ if and only if } m \leqslant n \text{ for all } m, n \in \mathbb{N} \text{ and } n \leqslant 0 \text{ for all } n \in \mathbb{N}.$$

Then the pairs $(\mathbb{N}_, \leqslant_1)$ and $(\mathbb{N}_0, \leqslant_2)$ are isomorphic to the ordinal numbers $\mathbb{N}_0$ and $\mathbb{N}_0 \cup \{\mathbb{N}_0\}$, respectively.

Let us call an ordinal number N **associated to a set** A if there exists a well ordering $\leqslant$ on the set A such that the pair $(A, \leqslant)$ is isomorphic to the ordinal number N. As we have seen above different ordinal numbers may be associated to the set A (even if the set A is an ordinal number itself).

We will analyze this situation in more detail in Unit *Cardinal Numbers* [Garden 2020i].

Historical Notes:

Theorem 7.7 (each well-ordered set is isomorphic to exactly one ordinal number) is due to von Neumann:

17. Ξ *sei wohlgeordnet. Dann gibt es eine und nur eine (durch Subsumption geordnete) Ordnungszahl, die dem* Ξ *ähnlich ist, nämlich* $\mathrm{OZ}(\Xi)$.

See [von Neumann 1922, p. 207].

17. *Let* Ξ *be well-ordered. Then there exists one and only one ordinal number (ordered by subsumption) similar to* Ξ*, namely* $\mathrm{OZ}(\Xi)$.

8 Notes and References

Do you want to learn more? In Unit *Finite Sets and their Cardinalities* [Garden 2020g] we have defined the cardinalities of finite sets. In Unit *Well-Ordered Sets* [Garden 2020h] we have studied the process of counting which led us to the well-ordered sets. In this unit we introduced the counting of well-ordered sets which led us to ordinal numbers. As we noticed in Remark 7.8 ordinal numbers are too fine to be cardinal numbers. In Unit *Cardinal Numbers* [Garden 2020i] you will learn how ordinal numbers can be used to define the cardinality of a set.

9 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Burali-Forti, Cesare (1897). "Una questione sui numeri transfiniti". In: *Rendiconti del circolo matematico di Palermo* 11, pp. 154–164. URL: https://doi.org/10.1007/BF03015911. See GDZ (cit. on pp. 23, 24).

Cantor, Georg (1895). "Beiträge zur Begründung der transfiniten Mengenlehre 1". In: *Mathematische Annalen* 46, pp. 481–512. See DigiZeitschriften (cit. on p. 13).

— (1897). "Beiträge zur Begründung der transfiniten Mengenlehre 2". In: *Mathematische Annalen* 49, pp. 207–246. See DigiZeitschriften (cit. on pp. 13, 18).

von Neumann, John (1922). "Zur Einführung der transfiniten Zahlen". In: *Acta litterarum ac scientiarum Regiae Universitatis Hungaricae Francisco-Josephinae: Sectio scientiarum mathematicarum* 1, pp. 199–208. This article is published under the name Neumann, János. (Cit. on pp. 13, 14, 18, 19, 21, 28).

10 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *Unions and Intersections of Sets*. URL: https://www.math-garden.com/unit/nst-unions (cit. on p. 18).

— (2020b). *Functions and Equivalent Sets*. URL: https://www.math-garden.com/unit/nst-functions (cit. on p. 16).

— (2020c). *Families and the Axiom of Choice*. URL: https://www.math-garden.com/unit/nst-families (cit. on p. 14).

— (2020d). *Ordered Sets and the Lemma of Zorn*. URL: https://www.math-garden.com/unit/nst-ordered-sets (cit. on pp. 7–9, 26).

— (2020e). *Successor Sets and the Axioms of Peano*. URL: https://www.math-garden.com/unit/nst-successor-sets (cit. on pp. 7, 9).

— (2020f). *The Natural Numbers and the Principle of Induction*. URL: https://www.math-garden.com/unit/nst-natural-numbers (cit. on pp. 7, 9, 10).

— (2020g). *Finite Sets and their Cardinalities*. URL: https://www.math-garden.com/unit/card-finite-sets (cit. on pp. 3, 28).

— (2020h). *Well-Ordered Sets*. URL: https://www.math-garden.com/unit/card-well-ordered (cit. on pp. 3, 4, 7, 10–12, 15, 17, 24, 26, 28).

— (2020i). *Cardinal Numbers*. URL: https://www.math-garden.com/unit/card-cardinal (cit. on pp. 3, 4, 28).

— (2020j). *Cardinal Arithmetic*. URL: https://www.math-garden.com/unit/card-arithmetic (cit. on p. 3).

— (2020k). *The Axiomatics of von Neumann, Bernays and Gödel*. In preparation (cit. on p. 3).

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