M. Garden

Finite Sets and their Cardinalities

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1 Introduction

The present unit is part of the walk *The Cardinality of Sets* consisting of the following units:

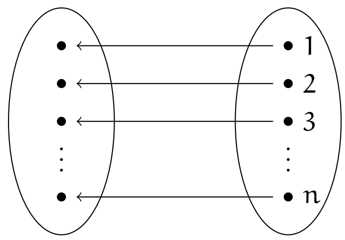
1. Finite Sets and their Cardinalities (this unit)
2. Well-Ordered Sets [Garden 2020f]
3. Ordinal Numbers [Garden 2020g]
4. Cardinal Numbers [Garden 2020h]
5. Cardinal Arithmetic [Garden 2020i]
6. The Axiomatics of von Neumann, Bernays and Gödel [Garden 2020j]

*

In the present unit we will explain the cardinality of finite sets. The cardinality of infinite sets will be explained in Unit *Cardinal Numbers* [Garden 2020h].

Finite Sets and their Cardinalities (see Section 2):

The idea of a finite set is quite obvious: We say that a set A is a **finite set** with n elements for a natural number n if there exists a bijective mapping $\alpha : \{1, \ldots, n\} \to A$ from the set $\{1, \ldots, n\}$ onto the set A (Definition 2.4). In this case we write $|A| = n$.



Obviously, the definition $|A| = n$ requires that the equations $|A| = m$ and $|A| = n$ imply that $m = n$. This is shown in Theorem 2.3.

An elementary but important result is the fact that a subset of a finite set is finite (Theorem 2.10). A further elementary but important result says that every finite ordered set has a minimal and a maximal element (Theorem 2.14). We will use this result in the proof of Theorem 2.16 saying that a set A is infinite if and only if for each natural number n there exists a subset A_n of the set A with exactly n elements.

Finally, we will show in Theorem 2.17 that the set $\mathbb{N}_0$ is infinite.

Sums, Products, Powers and Finite Sets (see Section 3:)

In Unit *Unions and Intersections of Sets* [Garden 2020a] we have introduced the following elementary operations on sets: Union of sets, intersection of sets, complement of a set and direct product of sets. We are now going to explain the relation between the cardinality and these operations on (finite) sets. These results are all quite elementary.

Given two finite disjoint sets A and B we have $|A \cup B| = |A| + |B|$ (Theorem 3.1).

Given two arbitrary finite sets A and B we have $|A| + |B| = |A \cup B| + |A \cap B|$ (Theorem 3.3).

Given a finite set A and a subset B of the set A we have $|A| = |A \setminus B| + |B|$ (Theorem 3.2).

Given two finite sets A and B we have $|A \times B = |A| \cdot |B|$. Given n finite sets $A_1, \ldots, A_n$ we have

$$\left|\prod_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|.$$

See Theorem 3.9.

In this context it would fit very well to compute the cardinality of the power set $\mathcal{P}(A)$ of a finite set A and the number of subsets with k elements of a set A with n elements. Given a set A with n elements and a natural number k we have

$$\begin{aligned} |\mathcal{P}(A)| &= 2^n \text{ and} \\ |\{B \subseteq A \mid |B| = k\}| &= \binom{n}{k} \text{ (speak: } n \text{ choose } k). \end{aligned}$$

This will be explained in Unit *The Binomial Coefficients and the Triangle of Pascal* [Garden 2020l].

In Unit *Functions and Equivalent Sets* [Garden 2020c] we have defined the sets $\mathcal{F}(B, A)$ and $\mathcal{B}(A)$ consisting of the functions $f : B \to A$ from a set B into a set A and of the bijective functions $f : A \to A$ from a set A onto itself, respectively. We will compute the cardinalities of these sets: Let A and B be two finite sets with $|A| = m$ and $|B| = n$ for two natural numbers m and n. Then we have

$$|\mathcal{F}(B, A)| = m^n \text{ and } |\mathcal{B}| = n!$$

where $n! := 1 \cdot \ldots \cdot n$ (speak: factorial of n) is explained in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] (see Theorem 3.12 and Theorem 3.14).

Injective and Surjective Mappings between Finite Sets (see Section 4):

We conclude this unit with the remarkable result that a mapping $\alpha : A \to B$ from a finite set A into a finite set B is bijective if and only if it is injective if and only if it is surjective (Theorem 4.2).

2 Finite Sets and their Cardinalities

Equivalent Sets:

We first recall the definition (Definition 2.1) and some elementary properties (Proposition 2.2) of equivalent sets. They are explained in detail in Unit *Functions and Equivalent Sets* [Garden 2020c].

2.1 Definition. Two sets A and B are called **equivalent** if there exists a bijective function $f : A \to B$ from the set A onto the set B. If the sets A and B are equivalent, we write $A \sim B$.[a]

[a] For the definition of bijective functions see Unit *Functions and Equivalent Sets* [Garden 2020c].

2.2 Proposition. *Let A, B and C be three sets.*

(a) We have $A \sim A$ (reflexivity).

(b) If $A \sim B$, then we have $B \sim A$ (symmetry).

(c) If $A \sim B$ and $B \sim C$, then we have $A \sim C$ (transitivity).

Proof. For the proof see Unit *Functions and Equivalent Sets* [Garden 2020c]. $\square$

The Definition of Finite and Infinite Sets:

We first recall that we have defined the natural numbers in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] as specific sets, namely

$$0 := \emptyset,\ 1 := \{0\},\ 2 := \{0, 1\}, \ldots$$

and so on. In particular, we have

$$n + 1 = \{0, 1, \ldots, n\}$$

for all natural numbers n.

We need the following elementary result for the definition of finite sets:

2.3 Theorem. *Let m and n be two natural numbers. If the sets m and n are equivalent, then we have $m = n$.*

Proof. Step 1. *Let m and n be two equivalent numbers such that $m \leqslant n$. Then we have $m = n$:*

We proceed by induction on m:

$m = 0$: Let n be a natural number such that $m \sim n$ and $m \leqslant n$. Since $m = 0 = \emptyset$, we have

$$m \sim n \text{ if and only if } n = \emptyset.$$

It follows that $m = n$.

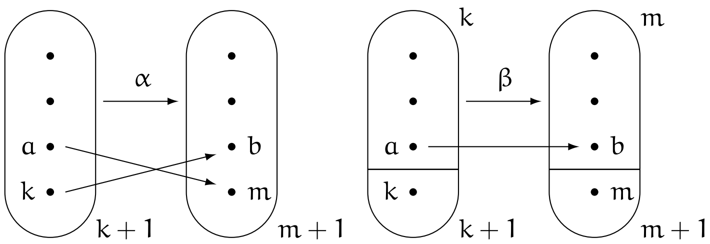
$m \mapsto m+1$: Let n be a natural number such that $n \sim m+1$ and $m+1 \leqslant n$. Since

$$n \geqslant m + 1 > 0,$$

it follows that $n > 0$. Hence, there exists a natural number k such that $n = k + 1$. It follows that $k+1 \sim m+1$. Note that $k+1 = k \uplus \{k\}$ and that $m+1 = m \uplus \{m\}$. Since $k+1 \sim m+1$, there exists a bijective function $\alpha : k+1 \longrightarrow m+1$.

Let $a := \alpha^{-1}(m)$ and $b := \alpha(k)$. Define the function $\beta : k \longrightarrow m$ by

$$\beta(x) := \begin{cases} \alpha(x) & \text{if} \quad x \neq a \\ b & \text{if} \quad x = a \end{cases}$$



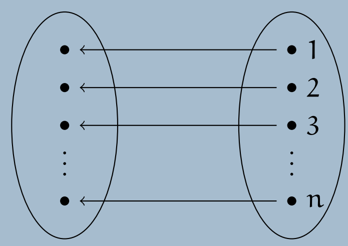
Since the function $\alpha : k+1 \to m+1$ is bijective, the function $\beta : k \to m$ is also bijective. It follows that $m \sim k$. Since $m+1 \leqslant n = k+1$, it follows that $m \leqslant k$. By induction, we get $m = k$ implying that $m+1 = k+1 = n$.

Step 2. *Let m and n be two equivalent numbers. Then we have $m = n$:*

Since the set $\mathbb{N}_0$ is totally ordered, we have $m \leqslant n$ or $n \leqslant m$. In both cases, it follows from Step 1 that $m = n$. □

2.4 Definition. Let A be a set.

(a) The set A is called **finite** if we have $A = \emptyset$ or if there exists a natural number $n \geqslant 1$ such that the sets A and $\{1, \ldots, n\}$ are equivalent.
(b) If $A = \emptyset$, then we say that the set A has 0 elements, and we write $|A| = 0$.



(c) If the sets A and $\{1, \ldots, n\}$ are equivalent for some natural number $n \geqslant 1$, then we say that the set A has n elements, and we write $|A| = n$.

(d) For a finite set A the value $|A|$ is called the **cardinality** of the set A. It is also denoted by $\operatorname{card}(A)$.

(e) If the set A is finite, then we also write $|A| < \infty$.

(f) If the set A is not finite, then the set A is called **infinite**. In this case we write $|A| = \infty$.

French / German. Finite set = Ensemble fini = Endliche Menge. Cardinality = Cardinalité = Mächtigkeit. Infinite Set = Ensemble infini = Uendliche Menge.

2.5 Remarks. (a) Note that it follows from Theorem 2.3 that the definition $|A| = n$ is well-defined in the sense that $|A| = m$ and $|A| = n$ implies that $m = n$.

(b) The situation is different for the case $|A| = \infty$. In this case it does not follow from $|A| = \infty$ and $|B| = \infty$ that the sets A and B are equivalent. For example, we have $|\mathbb{N}_0| = \infty$ and $|\mathbb{R}| = \infty$ for the set $\mathbb{N}_0$ of the natural numbers and the set $\mathbb{R}$ of the real numbers, but there does not exist a bijective mapping from the set $\mathbb{N}_0$ onto the set $\mathbb{R}$. This important theorem is due to Georg Cantor. The details are explained in Unit *The Real Numbers* [Garden 2020m]. In Unit *Cardinal Numbers* [Garden 2020h] we will explain the cardinality of infinite sets in more detail.

2.6 Examples. We have

$$|\{4\}| = 1, |\{1,2,3\}| = |\{2,6,9\}| = 3 \text{ and } |\{x,y\}| = 2 \text{ if } x \neq y.$$

Elementary Properties of Finite Sets:

2.7 Proposition. *Let A be a set.*

(a) We have $|A| = 0$ *if and only if* $A = \emptyset$.

(b) We have $|A| = 1$ *if and only if there exists an element* a *of the set* A *such that* $A = \{a\}$.

Proof. (a) follows from Definition 2.4.

(b) Suppose that $|A| = 1$. Then there exists a bijective mapping $\alpha : \{1\} \to A$ from the set $\{1\}$ onto the set A implying that $A = \{\alpha(1)\}$.

Conversely, suppose that $A = \{a\}$ for an element a of the set A. Then the mapping

$$\alpha : \{1\} \to A, \ \alpha : 1 \mapsto a$$

is obviously a bijective mapping from the set $\{1\}$ onto the set $A = \{a\}$ implying that $|A| = 1$. □

2.8 Proposition. *Let* A *be a finite set with* n *elements for some natural number* n. *Then the sets* A *and* n *are equivalent. In particular, we have* $|n| = n$.

Proof. If $n = 0$, then we have $A = \emptyset$ (Proposition 2.7) and $n = \emptyset$. Hence, we have $A = n$. In particular, the sets A and n are equivalent.

Let $n \geqslant 1$. Then there exists a natural number m such that $n = m + 1$. It follows that

$$n = m + 1 = \{0, 1, \dots, m\}.$$

Let

$$\alpha : \{0, 1, \dots, m\} \to \{1, \dots, m + 1\} = \{1, \dots, n\}$$

be the mapping defined by $\alpha : x \mapsto x + 1$.

The mapping α is obviously injective.

We claim that the mapping α is surjective: For, let y be an element of the set

$$\{1, \dots, n\} = \{1, \dots, m + 1\} = \{z \in \mathbb{N}_0 \mid 1 \leqslant z \leqslant m + 1\}.$$

Since $y \geqslant 1$, there exists a natural number x such that $y = x + 1$. It follows from

$$x + 1 = y \leqslant m + 1$$

that $x \leqslant m$. Hence, the natural number x is contained in the set

$$\{z \in \mathbb{N}_0 \mid 0 \leqslant z \leqslant m\} = \{1, \dots, m\},$$

and we have $\alpha(x) = x + 1 = y$.

It follows that the mapping $\alpha : \{0, 1, \dots, m\} \to \{1, \dots, n\}$ is bijective. Hence, the sets $n = \{0, 1, \dots, m\}$ and $\{1, \dots, n\}$ are equivalent. Since $|A| = n$, the sets A and $\{1, \dots, n\}$ are equivalent. It follows from Proposition 2.2 that the sets A and n are equivalent.

Since the sets A and n are equivalent, we have

$$n = |A| = |n|.$$

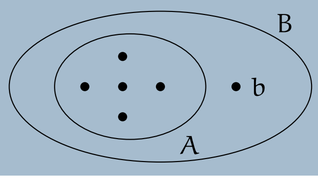
□

Subsets of Finite Sets:

2.9 Theorem. *Let A be a set, let b be an element not contained in the set A, and let $B := A \mathbin{\dot\cup} \{b\}$.*

Then the set A is finite if and only if the set B is finite. In this case, we have

$$|B| = |A| + 1.$$



Proof. If the set A is empty, then we have $B = \{b\}$, and it follows from Proposition 2.7 that

$$|B| = 1 = 0 + 1 = |A| + 1.$$

Hence, we may suppose that $A \neq \emptyset$.

Step 1. *If the set A is finite, then the set B is finite, and we have $|B| = |A| + 1$:*

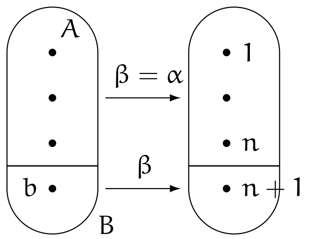
Since the set A is finite, there exists a natural number n such that $|A| = n$. By Definition 2.4, there exists a bijective function $\alpha : A \to \{1, \dots, n\}$ from the set A onto the set $\{1, \dots, n\}$.

Therefore, the function

$$\beta : B = A \mathbin{\dot\cup} \{b\} \to \{1, \dots, n, n+1\}$$

defined by

$$\beta(x) := \begin{cases} \alpha(x) & \text{if} \quad x \in A \\ n+1 & \text{if} \quad x = b \end{cases}$$



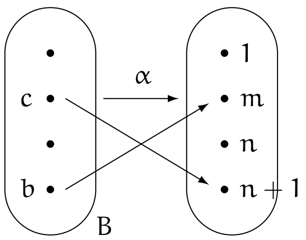
is bijective, and it follows that $|B| = n + 1$.

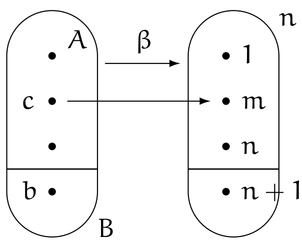
Step 2. *If the set B is finite, then the set A is finite, and we have $|B| = |A| + 1$:*

Since $B \neq \emptyset$, we have $|B| > 0$ (Proposition 2.7). Hence, there exists a natural number n such that $|B| = n + 1$. It follows that there exists a bijective function $\alpha : B \to \{1, \dots, n, n+1\}$ from the set B onto the set $\{1, \dots, n, n+1\}$. Let $m := \alpha(b)$, and let c be the element of the set B such that $\alpha(c) = n + 1$.

Define the function $\beta : A \to \{1, \dots, n\}$ by

$$\beta(x) := \begin{cases} \alpha(x) & \text{if} \quad x \neq c \\ m & \text{if} \quad x = c \end{cases}$$





It follows that the function $\beta : A \to \{1, \dots, n\}$ is bijective, and we get

$$|B| = n + 1 = |A| + 1.$$

$\square$

2.10 Theorem. *Let B be a finite set, and let A be a subset of the set B.*

(a) The set A is finite, and we have $|A| \leqslant |B|$.

(b) We have

$$|A| = |B| \text{ if and only if } A = B.$$

Proof. (a) Let $|B| = n$ for a natural number n. We proceed by induction on n:

$n = 0$: It follows from $|B| = n = 0$ that $B = \emptyset$ implying that $A = \emptyset$. Hence, the set A is finite, and we have

$$|A| = 0 \leqslant 0 = |B|.$$

$n \mapsto n + 1$: Suppose that we have $|B| = n + 1$.

Case 1. *Suppose that* $A = B$.

Then the set $A = B$ is obviously finite, and we have $|A| \leqslant |B|$.

Case 2. *Suppose that* $A \neq B$.

Then the set A is a proper subset of the set B, and there exists an element x of the set B not contained in the set A. Let $B' := B \setminus \{x\}$. Since $B = B' \,\dot{\cup}\, \{x\}$, it follows from Theorem 2.9 that the set B' is finite and that $|B'| = n$. Since the element x is not contained in the set A, the set A is a subset of the set B'. By induction, it follows that the set A is finite and that

$$|A| \leqslant |B'| = n.$$

In particular, we have $|A| \leqslant n \leqslant n + 1 = |B|$.

(b) Step 1. *Suppose that* $A = B$. *Then we have* $|A| = |B|$*:*

The assertion is obvious.

Step 2. *Suppose that* $|A| = |B|$. *Then we have* $A = B$*:*

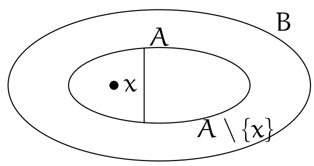
Let $n := |A| = |B|$. We proceed by induction on n:

$n = 0$: If $n = 0$, then we have $A = \emptyset$ and $B = \emptyset$ implying that $A = B$.

$n \mapsto n + 1$: Suppose that $|A| = |B| = n + 1$. Since $|A| = n + 1 > 0$, there exists an element x of the set A. Since the set A is a subset of the set B, it follows that the element x is also contained in the set B.

Let $A' := A \setminus \{x\}$ and $B' := B \setminus \{x\}$. It follows from Theorem 2.9 that

$$|A'| = |B'| = n.$$



Since the set A is a subset of the set B, the set $A' = A \setminus \{x\}$ is a subset of the set $B' = B \setminus \{x\}$. By induction, we get $A' = B'$. It follows that

$$A = A' \cup \{x\} = B' \cup \{x\} = B.$$

□

2.11 Theorem. *Let A and B be two finite sets. If the set A is a proper subset of the set B, then we have $|A| < |B|$. In particular, the sets A and B are not equivalent.*

Proof. The assertion follows from Theorem 2.10. □

2.12 Remark. We will show in Unit *Cardinal Numbers* [Garden 2020h] that a set A is infinite if and only if there exists a bijective mapping $\alpha : A \to B$ from the set A onto a proper subset B of the set A.

2.13 Theorem. *Let A be a finite set with $|A| = n$ for a natural number n, and let m be a further natural number with $m \leqslant n$.*

Then there exists a subset B of the set A such that $|B| = m$.

Proof. Since $|A| = n$, there exists a bijective mapping

$$\alpha : \{1, \ldots, n\} \to A$$

from the set $\{1, \ldots, n\}$ onto the set A. Since $m \leqslant n$, the set $\{1, \ldots, m\}$ is a subset of the set $\{1, \ldots, n\}$. It follows that the set

$$B := \big\{\alpha(1), \ldots, \alpha(m)\big\}$$

is a subset of the set A with $|B| = m$. □

Minimal and Maximal Elements in Finite Ordered Sets:

Ordered sets are explained in Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. We understand by an ordered set $A = (A, \leqslant)$ a *partially* ordered set, that is, there may exist elements x and y of the set A such that neither $x \leqslant y$ nor $y \leqslant x$.

2.14 Theorem. *Let $A = (A, \leqslant)$ be a non-empty finite (partially) ordered set. Then the set A has a maximal and a minimal element.*

Proof. Since the set A is non-empty and finite, there exists a natural number $n \geqslant 1$ such that $|A| = n$. W.l.o.g. we may assume that

$$A = \{a_1, \ldots, a_n\}.$$

We proceed by induction on n:

$n = 1$: Obviously, the element a_1 is the minimal and the maximal element of the set $A = \{a_1\}$.

$n \mapsto n + 1$: Let $A = \{a_1, \ldots, a_n, a_{n+1}\}$. By induction, the set $A_n := \{a_1, \ldots, a_n\}$ has a minimal element b and a maximal element c.

If $a_{n+1} < b$, then the element a_{n+1} is a minimal element of the set A. Otherwise, the element b is a minimal element of the set A.

If $a_{n+1} > c$, then the element a_{n+1} is a maximal element of the set A. Otherwise, the element c is a maximal element of the set A. □

2.15 Example. Let

$$A := \big\{\{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}\big\}$$

ordered by the subset relation $\subseteq$. Then the set $A = (A, \subseteq)$ is a partially ordered set. The minimal elements are the sets

$$\{1\}, \{2\}, \{3\}.$$

The maximal elements are the sets

$$\{1,2\},\{1,3\},\{2,3\}.$$

Infinite Sets:

2.16 Theorem. *Let A be a set. Then the following conditions are equivalent:*

(i) The set A is infinite, that is, $|A| = \infty$.

(ii) For each natural number n there exists a finite subset A_n of the set A such that $|A_n| = n$.

(iii) For each natural number n there exists a finite subset A_n of the set A such that $|A_n| \geqslant n$.

Proof. (i) $\Rightarrow$ (ii): Assume that there exists a natural number n such that

$$|X| \neq |n| \text{ for all } X \in \mathcal{P}(A).$$

Let

$$N := \{m \in \mathbb{N}_0 \mid m \leqslant n, \exists\, B \in \mathcal{P}(A) : |B| = m\}$$

be the set of the natural numbers $m \leqslant n$ such that there exists a subset B of the set A with $|B| = m$. Since

$$N \subseteq \{0,1,\ldots,n\} \text{ and } 0 \in N,$$

it follows from Theorem 2.10 that the set N is non-empty and finite. By Theorem 2.14, the set N has a maximal element r.

Let R be a subset of the set A with $|R| = r$. We claim that $R = A$:

Otherwise, there exists an element x of the set A which is not contained in the set R. It follows from Theorem 2.9 that the set $R \uplus \{x\}$ is finite with

$$|R \uplus \{x\}| = |R| + 1 = r + 1.$$

On the other hand, since $r + 1 \leqslant n$ and since r is the maximal element of the set N, it follows that the set $R \uplus \{x\}$ is no subset of the set A, a contradiction.

It follows from $R = A$ that

$$|A| = |R| = r < \infty,$$

a contradiction.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i): Assume that the set A is finite. Then there exists a natural number n such that $|A| = n$. On the other hand, it follows from Condition (iii) that $|A| \geqslant n+1$, a contradiction. □

2.17 Theorem. *The set $\mathbb{N}_0$ of the natural numbers is infinite.*

Proof. Let n be a natural number. Then the set $\{1,\ldots,n\}$ is a finite set with exactly n elements contained in the set $\mathbb{N}_0$. It follows from Theorem 2.16 that the set $\mathbb{N}_0$ is infinite. □

2.18 Remark. An alternative proof of Theorem 2.17 is as follows: The mapping $\alpha : \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto n+1$ is bijective implying that the set $\mathbb{N}_0$ is equivalent to one of its proper subsets. As we have seen in Theorem 2.11, this situation cannot occur for finite sets.

In Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] you will find a section about the definition of the natural numbers according to Richard Dedekind. His approach is based on the definition that a set A is called **infinite** if there exists a bijective mapping $\alpha : A \to B$ from the set A onto a proper subset B of the set A.

Historical Notes:

The idea of counting the number of the elements of a set A via a bijective mapping from this set A onto a reference system is very old:

One example is the so-called Ishango bone which has the special feature of having a series of notches. Even if the meaning of these notches is not clear, it is believed that this bone or similar bones were used for counting. If a flock of sheep has 30 animals and a bone has 30 notches, the shepherd can check the completeness of his flock by comparing the number of notches with the number of animals. He doesn't even have to know the number 30 to do this. The Ishango bone is probably more than 20.000 years old.

In other words, the shepherd uses a bijective mapping between his flock and the set of the notches of is bone.

Theorem 2.3 (if two natural numbers m and n are equivalent, then we have $m = n$) is obvious in view of an intuitive understanding of the natural numbers. It says that two different natural numbers describe different cardinalities.

However, an axiomatic definition of the natural numbers requires a proof of this property, or, as Richard Dedekind said:

Was beweisbar ist, soll in der Wissenschaft nicht ohne Beweis geglaubt werden.

See [Dedekind 1932, p. 335].

In science, what can be proven should not be believed without proof.

Of course, the proof stems from Dedekind:

120. Satz. *Sind m, n verschiedene Zahlen, so sind Z_m, Z_n unähnliche Systeme.*

See [Dedekind 1932, p. 335].

120. Satz. *If m and n are different numbers, then Z_m and Z_n are non-equivalent sets.*

Note that

$$Z_n := \{x \in \mathbb{N} \mid x \leqslant n\} \text{ for all } n \in \mathbb{N}.$$

Theorem 2.9 is also due to Dedekind:

166. Satz. *Ist $\Gamma = \mathcal{M}(B, \gamma)$, wo B ein System von n Elementen und γ ein nicht in B enthaltenes Element von Γ bedeutet, so besteht Γ aus n' Elementen.*

See [Dedekind 1932, p. 388].

166. Satz. *If $\Gamma = \mathcal{M}(B, \gamma)$ where B is a set of n elements and γ is an element of Γ not contained in B, then Γ consists of n' elements.*

In modern terminology $\Gamma = \mathcal{M}(B, \gamma)$ means $\Gamma = B \cup \{\gamma\}$, and n' means $n + 1$ (successor of n).

Theorem 2.10 is also contained in [Dedekind 1932]:

165. Satz. *Ist* T *ein echter Teil eines endlichen Systems* Σ*, so ist die Anzahl der Elemente von* T *kleiner als diejenige der Elemente von* Σ*.*

See [Dedekind 1932, p. 388].

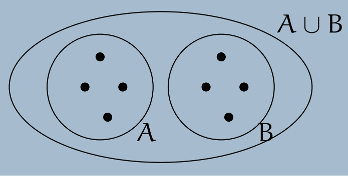
165. Satz. *If* T *is a proper subset of a set* Σ*, then the number of elements of* T *is smaller than the number of elements of* Σ*.*

3 Sums, Products, Powers and Finite Sets

3.1 Theorem. *Let* A *and* B *be two finite sets.*

If the sets A *and* B *are disjoint, then the set* $A \dot{\cup} B$ *is finite, and we have*

$$|A \dot{\cup} B| = |A| + |B|.$$



Proof. Let $|A| = m$ and $|B| = n$ for two natural numbers m and n. We proceed by induction on m:

$m = 0$: It follows from $m = 0$ that $A = \emptyset$. Hence, we have

$$|A \dot{\cup} B| = |B| = n = 0 + n = |A| + |B|.$$

$m \mapsto m + 1$: Suppose that $|A| = m + 1$ and $|B| = n$. Then we have $A \neq \emptyset$, and there exists an element a of the set A. Let $A' := A \setminus \{a\}$. By Theorem 2.9, the set A' is finite and we have

$$m + 1 = |A| = |A' \dot{\cup} \{a\}| = |A'| + 1$$

implying that $|A'| = m$. Since

$$A' \cap B \subseteq A \cap B = \emptyset,$$

it follows by induction that the set $A' \dot{\cup} B$ is finite and that

$$|A' \dot{\cup} B| = |A'| + |B| = m + n.$$

Finally, we have

$$A \cup B = (A' \cup B) \dot{\cup} \{a\},$$

and it follows from Theorem 2.9 that the set $A \cup B$ is finite and that

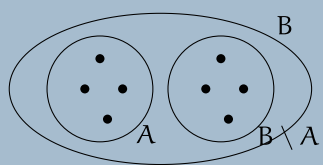
$$|A \cup B| = |A' \cup B| + 1 = m + n + 1 = (m + 1) + n.$$

□

3.2 Theorem. *Let* B *be a finite set, and let* A *be a subset of the set* B*.*

Then the sets A *and* $B \setminus A$ *are finite, and we have*

$$|B| = |A| + |B \setminus A|.$$



Proof. It follows from Theorem 2.10 that the sets A and $B \setminus A$ are finite. Since

$$B = A \mathbin{\dot{\cup}} B \setminus A,$$

it follows from Theorem 3.1 that

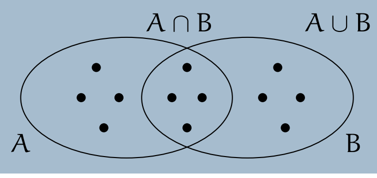
$$|B| = |A| + |B \setminus A|.$$

□

3.3 Theorem. *Let A and B be two finite sets.*

Then the union $A \cup B$ is finite, and we have

$$|A| + |B| = |A \cup B| + |A \cap B|.$$



Proof. By Theorem 3.2, the sets $A \cap B$ and $B \setminus (A \cap B)$ are finite, and we have

$$|B| = |A \cap B| + |B \setminus (A \cap B)|.$$

Since we have

$$A \cup B = A \mathbin{\dot{\cup}} \big(B \setminus (A \cap B)\big),$$

it follows from Theorem 3.1 that the set $A \cup B$ is finite and that we have

$$|A \cup B| = |A| + |B \setminus (A \cap B)|.$$

It follows that

$$|A| + |B| = |A| + |A \cap B| + |B \setminus (A \cap B)| = |A \cup B| + |A \cap B|.$$

□

3.4 Example. Let $A := \{1, 2, 3\}$ and $B := \{2, 3, 4, 5\}$. Then we have

$$\begin{aligned} |A| + |B| &= 3 + 4 = 7 \text{ and} \\ |A \cup B| + |A \cap B| &= 5 + 2 = 7. \end{aligned}$$

3.5 Remark. In Unit *The Integers* [Garden 2020k] we will introduce the integers and in particular the negative numbers. Then Theorem 3.3 can be formulated in the more usual form

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

3.6 Proposition. *Let $\emptyset \neq A$ be a finite set of natural numbers such that $|A| = n$ for a natural number $n \geqslant 1$. Let*

$$\alpha : \{1, \ldots, n\} \to A \text{ and } \beta : \{1, \ldots, n\} \to A$$

be two bijective mappings from the set $\{1, \ldots, n\}$ *onto the set* A. *Then we have*

$$\sum_{i=1}^{n} \alpha(i) = \sum_{i=1}^{n} \beta(i).$$

Proof. The assertion follows from the generalized associative law and the generalized commutative law explained in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e]. □

3.7 Definition. Let A be a finite set of natural numbers such that $|A| = n$ for a natural number n.

(a) If $A = \emptyset$, that is, if $n = 0$, then we set

$$\sum_{x \in A} x := 0.$$

(b) Suppose that $n \geqslant 1$, let $\alpha : \{1, \ldots, n\} \to A$ be a bijective mapping from the set $\{1, \ldots, n\}$ onto the set A, and let

$$a_i := \alpha(i) \text{ for all } i = 1, \ldots, n.$$

Then we set

$$\sum_{x \in A} x := \sum_{i=1}^{n} \alpha(i) = \sum_{i=1}^{n} a_i.$$

Note that, in view of Proposition 3.6, this definition is well-defined, that is, it is independent from the choice of the bijective function $\alpha : \{1, \ldots, n\} \to A$.

Products and the Direct Product of Sets:

For the proof of Theorem 3.9 we will need the following properties of direct products explained in Unit *Direct Products and Relations* [Garden 2020b]:

3.8 Proposition. *(a) We have*

$$X \times \emptyset = \emptyset \text{ for all sets } X.$$

(b) Let A, B *and* C *be three sets. Then we have*

$$\begin{aligned} A \times (B \cup C) &= (A \times B) \cup (A \times C), \text{ and} \\ A \times (B \cap C) &= (A \times B) \cap (A \times C). \end{aligned}$$

(c) Let A, B *and* C *be three sets such that* $B \cap C = \emptyset$. *Then we have*

$$A \times (B \,\dot{\cup}\, C) = (A \times B) \,\dot{\cup}\, (A \times C).$$

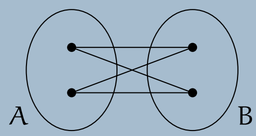
Proof. (a) and (b) are proven in Unit *Direct Products and Relations* [Garden 2020b].

(c) follows from (a) and (b). □

3.9 Theorem. *(a) Let A and B be two finite sets.*

Then the set $A \times B$ is finite, and we have

$$|A \times B| = |A| \cdot |B|.$$



(b) Let $A_1, \ldots, A_n$ be n finite sets. Then we have

$$\left|\prod_{i=1}^{n} A_i\right| = \prod_{i=1}^{n} |A_i|.$$

Proof. (a) Let $|A| = m$ and $|B| = n$ for two natural numbers m and n.

We proceed by induction on n:

$n = 0$: If $n = 0$, then $B = \emptyset$, and it follows from Proposition 3.8 that

$$|A \times B| = |\emptyset| = 0 = m \cdot 0 = m \cdot n.$$

$n \mapsto n+1$: Let C be a finite set such that $|C| = n+1$. Then we have $C \neq \emptyset$, and there exists an element c of the set C. Let $B := C \setminus \{c\}$.

It follows from Theorem 2.9 that B is a finite set with $|B| = n$. Since $C = B \,\dot\cup\, \{c\}$, it follows from Proposition 3.8 that

$$A \times C = A \times (B \,\dot\cup\, \{c\}) = (A \times B) \,\dot\cup\, (A \times \{c\}).$$

Since the mapping $\alpha : A \to A \times \{c\}$, $\alpha : x \mapsto (x, c)$ is obviously bijective, we have

$$m = |A| = |A \times \{c\}|.$$

By induction, we have $|A \times B| = m \cdot n$. Hence, it follows from Theorem 3.1 that

$$\begin{aligned} |A \times C| &= \big|(A \times B) \,\dot\cup\, (A \times \{c\})\big| \\ &= |A \times B| + |A \times \{c\}| \\ &= m \cdot n + m = m \cdot (n+1). \end{aligned}$$

(b) Since

$$\prod_{i=1}^{n+1} A_i = \left(\prod_{i=1}^{n} A_i\right) \times A_{n+1},$$

the assertion follows from (a) by induction. □

3.10 Example. Let $A := \{1, 2\}$. Then we have

$$A \times A = \{(1,1), (1,2), (2,1), (2,2)\}.$$

In particular, we have $|A| = 2$ and $|A \times A| = 2 \cdot 2 = 4$.

Exponentiation and Mappings between Sets:

For the proof of Theorem 3.12 we will need the following proposition:

3.11 Proposition. *Let $\alpha : A \to B$ be a mapping from a set A into a set B. If $A = \emptyset$, then we have $\alpha = \emptyset$.*

Proof. The mapping $\alpha : A \to B$ is a subset of the direct product $A \times B$. It follows from Proposition 3.8 that

$$\alpha \subseteq A \times B = \emptyset \times B = \emptyset.$$

Hence, we have $\alpha = \emptyset$. □

3.12 Theorem. *Let A and B be two finite sets with $|A| = m$ and $|B| = n$ for two natural numbers m and n. Then the set*

$$\mathcal{F}(B, A) := \left\{\alpha : B \to A \mid \alpha \text{ function}\right\}$$

is finite, and we have

$$|\mathcal{F}(B, A)| = m^n.$$

Proof. We proceed by induction on n:

$n = 0$: If $n = 0$, then $B = \emptyset$, and it follows from Proposition 3.11 that

$$|\mathcal{F}(B, A)| = \left|\left\{\alpha : \emptyset \to A \mid \alpha \text{ function}\right\}\right| = |\{\emptyset\}| = 1 = m^0 = m^n.$$

$n \mapsto n + 1$: Let C be a finite set such that $|C| = n + 1$. Then we have $C \neq \emptyset$, and there exists an element c of the set C. Let $B := C \setminus \{c\}$. It follows from Theorem 2.9 that B is a finite set with $|B| = n$.

Let $\alpha : B \to A$ be a mapping from the set B into the set A. For each element a of the set A let $\alpha_a : C \to A$ be the mapping defined by

$$\alpha_a(x) := \begin{cases} \alpha(x) & \text{if } x \in B \\ a & \text{if } x = c. \end{cases}$$

Then we have

$$\mathcal{F}(C, A) := \left\{\alpha_a : C \to A \mid \alpha \in \mathcal{F}(B, A) \text{ and } a \in A\right\}.$$

Let $\gamma : \mathcal{F}(B, A) \times A \to \mathcal{F}(C, A)$ be defined by $\gamma : (\alpha, a) \mapsto \alpha_a$. Then γ is a bijective mapping, and it follows from Theorem 3.9 and by induction that

$$|\mathcal{F}(C, A)| = |\mathcal{F}(B, A) \times A| = |\mathcal{F}(B, A)| \cdot |A| = m^n \cdot m = m^{n+1}.$$

□

3.13 Example. Let $A := \{1, 2\}$ and $B := \{a, b, c\}$. Then the set $\mathcal{F}(B, A)$ consists of the eight mappings $\alpha_i : B \to A$ defined as follows:

$$\begin{array}{ll} \alpha_1 : a \mapsto 1;\ b \mapsto 1;\ c \mapsto 1. & \alpha_2 : a \mapsto 1;\ b \mapsto 1;\ c \mapsto 2. \\ \alpha_3 : a \mapsto 1;\ b \mapsto 2;\ c \mapsto 1. & \alpha_4 : a \mapsto 1;\ b \mapsto 2;\ c \mapsto 2. \\ \alpha_5 : a \mapsto 2;\ b \mapsto 1;\ c \mapsto 1. & \alpha_6 : a \mapsto 2;\ b \mapsto 1;\ c \mapsto 2. \\ \alpha_7 : a \mapsto 2;\ b \mapsto 2;\ c \mapsto 1. & \alpha_8 : a \mapsto 2;\ b \mapsto 2;\ c \mapsto 2. \end{array}$$

We have $|A| = 2$, $|B| = 3$ and $|\mathcal{F}(B, A)| = 2^3 = 8$.

3.14 Theorem. *Let A be a finite set with n elements for a natural number n, and let $\mathcal{B}(A)$ be the set of the bijective functions from the set A onto itself. Then we have*

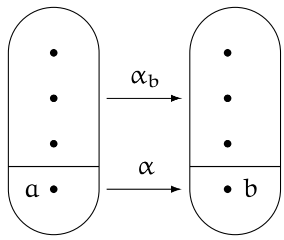
$$|\mathcal{B}(A)| = n!.$$

Proof. We proceed by induction on n: Note that $0! = 1$, $1! = 1$, $n! = 1 \cdot \ldots \cdot n$ and $(n+1)! = n! \cdot (n+1)$.

$n = 0$: If $n = 0$, then we have $A = \emptyset$, $\mathcal{B}(A) = \{\emptyset\}$ and therefore $|\mathcal{B}| = 1 = 0!$.

$n = 1$: If $n = 1$, then we have $A = \{a\}$ for an element a of the set A, the set $\mathcal{B}(A)$ consists of the function $\alpha : A \to A$, $\alpha : a \mapsto a$ and therefore $|\mathcal{B}| = 1 = 1!$.

$n \to n+1$ $(n \geqslant 1)$: Suppose that $|A| = n+1$, and let a be an element of the set A. Let $\alpha : A \to A$ be a bijective mapping from the set A onto itself, and let $b := \alpha(a)$. Then the mapping $\alpha : A \to A$ induces a bijective mapping $\alpha_b : A \setminus \{a\} \to A \setminus \{b\}$ from the set $A \setminus \{a\}$ onto the set $A \setminus \{b\}$ and vice versa.



By induction, it follows that

$$|\{\alpha \in \mathcal{B}(A) \mid \alpha(a) = b\}| = n!.$$

Since

$$\mathcal{B}(A) = \bigcup_{b \in A} \{\alpha \in \mathcal{B}(A) \mid \alpha(a) = b\} \text{ and } |A| = n+1,$$

it follows from Theorem 3.1 that

$$|\mathcal{B}(A)| = (n+1) \cdot n! = (n+1)!.$$

□

4 Injective and Surjective Mappings between Finite Sets

4.1 Theorem. *Let A and B be two finite sets, and let $\alpha : A \to B$ be a mapping from the set A into the set B.*

(a) Suppose that the mapping $\alpha : A \to B$ is injective. Then we have

$$|A| \leqslant |B| \textit{ and } |A| = |B| \textit{ if and only if } \alpha : A \to B \textit{ is bijective.}$$

(b) Suppose that the mapping $\alpha : A \to B$ is surjective. Then we have

$$|A| \geqslant |B| \textit{ and } |A| = |B| \textit{ if and only if } \alpha : A \to B \textit{ is bijective.}$$

Proof. (a) Let $B' := \alpha(A)$. Since the mapping $\alpha : A \to B$ is injective, the mapping $\alpha : A \to B' = \alpha(A)$ is bijective. Since the set B' is a subset of the set B, it follows from Theorem 2.10 that

$$|A| = |B'| \leqslant |B| \text{ and } |A| = |B'| = |B| \text{ if and only if } \alpha(A) = B' = B.$$

In the second case we have $\alpha(A) = B$, that is, the mapping $\alpha : A \longrightarrow B$ is surjective and therefore bijective.

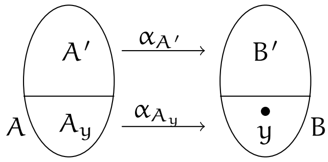
(b) We proceed by induction on $n := |B|$.

$n = 0$: If $n = 0$, then we have $B = \emptyset$. Obviously, we have $|A| \geqslant 0$ and $|A| = 0$ if and only if $A = \emptyset$. In the second case the mapping $\alpha : \emptyset \longrightarrow \emptyset$ is obviously bijective.

$n \mapsto n + 1$: Suppose that $|B| = n + 1$. Since $|B| = n + 1 > 0$, there exists an element y of the set B. Let $B' := B \setminus \{y\}$, and let

$$\begin{aligned} A' &:= \alpha^{-1}(B') = \{x \in A \mid \alpha(x) \in B'\} = \{x \in A \mid \alpha(x) \neq y\} \text{ and} \\ A_y &:= \alpha^{-1}\big(\{y\}\big) = \{x \in A \mid \alpha(x) = y\}. \end{aligned}$$

The mapping $\alpha : A \longrightarrow B$ splits into the two surjective mappings $\alpha|_{A'} : A' \longrightarrow B'$ and $\alpha|_{A_y} : A_y \longrightarrow \{y\}$. Since $|B'| = n$, it follows by induction that



$$|A'| \geqslant |B'| \text{ and } |A'| = |B'| \text{ if and only if } \alpha|_{A'} : A' \longrightarrow B' \text{ is bijective.}$$

Since $\alpha : A \longrightarrow B$ is surjective, we have

$$|A_y| \geqslant 1 \text{ and } |A_y| = 1 \text{ if and only if } \alpha|_{A_y} : A_y \longrightarrow \{y\} \text{ is bijective.}$$

Hence, we have

$$|A| = |A' \cup A_y| = |A'| + |A_y| \geqslant |B'| + 1 = |B|.$$

In addition, we have

$$\begin{aligned} |A| = |B| &\iff |A'| = |B'| \text{ and } |A_y| = 1 \\ &\iff \alpha|_{A'} : A' \longrightarrow B' \text{ and } \alpha|_{A_y} : A_y \longrightarrow \{y\} \text{ are bijective} \\ &\iff \alpha : A \longrightarrow B \text{ is bijective.} \end{aligned}$$

□

4.2 Theorem. *Let A and B be two finite sets, and let $\alpha : A \longrightarrow B$ be a mapping from the set A into the set B. If $|A| = |B|$, then the following conditions are equivalent:*

(i) The mapping $\alpha : A \longrightarrow B$ is injective.

(ii) The mapping $\alpha : A \longrightarrow B$ is surjective.

(iii) The mapping $\alpha : A \longrightarrow B$ is bijective.

Proof. The assertion follows from Theorem 4.1. □

4.3 Theorem. (pigeonhole principle) *Let $f : A \longrightarrow B$ be a function from a finite set A into a finite set B. If $|B| < |A|$, then the function $f : A \longrightarrow B$ is not injective, that is, there exist two elements x and y of the set A such that $f(x) = f(y)$.*

Proof. The assertion follows from Theorem 4.1. □

French / German. Pigeonhole Principle = Principe des Tiroirs = Schubfachprinzip.

4.4 Remark. The name pigeonhole principle comes from the following application: Suppose that you distribute 10 letters over nine pigeonholes. Then there exists at least one pigeonhole with at least two letters in it.

Historical Notes:

The following historical information about the pigeonhole principle stems from Rittaud and Heeffer [Rittaud, Benoît and Heeffer, Albrecht 2014]. It has been mentioned for the first time by Jean Leurechon in 1622:

Necesse est, duos hominum, habere totidem numero pilos, aureos, & similia.

See [Leurechon 1622, p. 2].

It is necessary that two men have the same number of hairs, gold, and others.

(Translation by Rittaud and Heeffer [Rittaud, Benoît and Heeffer, Albrecht 2014].)

Leurechon came back to this observation in his book Récréation mathématiques 1624:

Qu'il est totalement nécessaire que deux hommes aient autant de cheveux ou de pistolets l'un que l'autre.

See [Leurechon 1627, p. 222].

That it is absolutely necessary that two men have as many hairs or pistols as the other.

(Translation by Rittaud and Heeffer [Rittaud, Benoît and Heeffer, Albrecht 2014].)

The following explanations of Leurechon are based on the pigeonhole principle.

The first known application of the pigeonhole principle in mathematics is due to Peter Gustav Lejeune Dirichlet: He applies this principle in 1842 in two articles, one about continued fractions and a second about quadratic forms:

[...], so hat man $(2n+1)^m$ *echte Brüche, von denen notwenidg wenigstens zwei in demselben der durch die Werte*

$$0, \frac{1}{(2n)^m}, \frac{2}{(2n)^m}, \ldots, \frac{(2n)^m - 1}{(2n)^m}, 1$$

begrenzten $(2n)^m$ *Intervalle liegen müssen.*

See [Dirichlet 1842b, p. 94].

[...], then we have $(2n+1)^m$ *real fractions of which at least two must be contained in the same by the values*

$$0, \frac{1}{(2n)^m}, \frac{2}{(2n)^m}, \ldots, \frac{(2n)^m - 1}{(2n)^m}, 1$$

defined $(2n)^m$ *intervals.*

Or, comme avec de pareils entiers on ne peut former qu'un nombre de combinaisons distinctes, exprimé par $(2n)^2$*, tandis que celui des expressions* $\xi - a\eta$ *est* $(2n+1)^2$*, on voit que l'une au moins des cobinaisons* p*,* q *devra se reproduire.*

See [Dirichlet 1842a, p. 334].

Now, as with such integers we can only form a number of distinct combinations, expressed by $(2n)^2$*, while that of the expressions* $\xi - a\eta$ *is* $(2n+1)^2$*, we see that at least one of the combinations* p*,* q *must appear twice.*

The pigeonhole principle is also often called the *Principle of Dirichlet*. Sometimes one can read that Dirichlet has used this principle for the first time in 1834, but according to [Rittaud, Benoît and Heeffer, Albrecht 2014] no precise reference is known.

5 Notes and References

Do you want to learn more? In the next unit of the walk *The Cardinality of Sets*, Unit *Well Ordered Sets* [Garden 2020f], we will explain the famous well ordering theorem of Ernst Zermelo which is the basis for the definition of cardinal numbers.

6 Literature

For further literature about set theory please have a look at Literature about Set Theory.

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Rittaud, Benoît and Heeffer, Albrecht (2014). "The Pigeonhole Principle, two Decades before Dirichlet". In: *The Mathematical Intelligencer* 36.2, pp. 27–29. See Uiniv. of Gent (cit. on pp. 20, 21).

7 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *Unions and Intersections of Sets*. URL: `https://www.math-garden.com/unit/nst-unions` (cit. on p. 3).

— (2020b). *Direct Products and Relations*. URL: `https://www.math-garden.com/unit/nst-direct-products` (cit. on p. 15).

— (2020c). *Functions and Equivalent Sets*. URL: `https://www.math-garden.com/unit/nst-functions` (cit. on pp. 4, 5).

— (2020d). *Ordered Sets and the Lemma of Zorn*. URL: `https://www.math-garden.com/unit/nst-ordered-sets` (cit. on p. 10).

Garden, M. (2020e). *The Natural Numbers and the Principle of Induction*. URL: https://www.math-garden.com/unit/nst-natural-numbers (cit. on pp. 4, 5, 12, 15).

— (2020f). *Well-Ordered Sets*. URL: https://www.math-garden.com/unit/card-well-ordered (cit. on pp. 3, 21).

— (2020g). *Ordinal Numbers*. URL: https://www.math-garden.com/unit/card-ordinal (cit. on p. 3).

— (2020h). *Cardinal Numbers*. URL: https://www.math-garden.com/unit/card-cardinal (cit. on pp. 3, 6, 10).

— (2020i). *Cardinal Arithmetic*. URL: https://www.math-garden.com/unit/card-arithmetic (cit. on p. 3).

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