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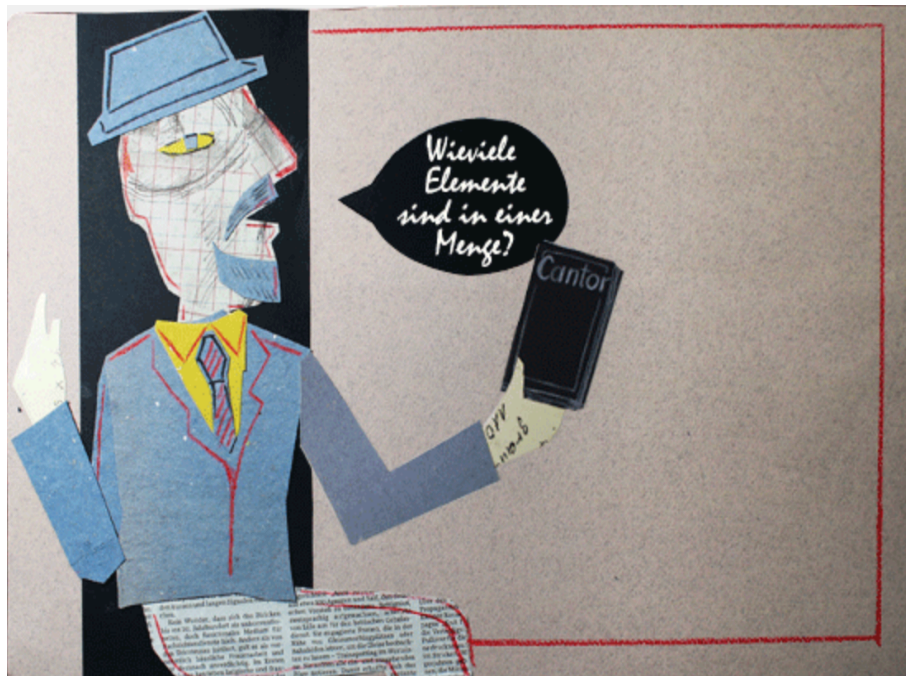
Cardinal Numbers

Impressum

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Contents

1   Introduction

The present unit is part of the walk *The Cardinality of Sets* consisting of the following units:

1. Finite Sets and their Cardinalities [Garden 2020h]

2. Well-Ordered Sets [Garden 2020i]

3. Ordinal Numbers [Garden 2020j]

4. Cardinal Numbers (this unit)

5. Cardinal Arithmetic [Garden 2020k]

6. The Axiomatics of von Neumann, Bernays and Gödel [Garden 2020l]

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When we count the elements of a finite set, we proceed as follows: We first define the natural numbers

$$0 := \emptyset, \ 1 := \{\emptyset\} = \{0\}, \ 2 := \{0, 1\}, \ldots, \ n + 1 := \{0, 1, 2, \ldots, n\}, \ldots$$

and then we say that a set A is **finite with exactly n elements** if there exists a bijective mapping $\alpha : A \to n$ from the set A onto the set n. In this case we write $|A| = n$. This procedure has been explained in detail in Unit *Finite Sets and their Cardinalities* [Garden 2020h].

This method of counting works very well since for any finite set A there exists exactly one natural number n such that $|A| = n$. In order to transfer this procedure to arbitrary sets, we need to generalize the natural numbers to the so-called cardinal numbers with the property that for each set A there exists exctly one cardinal number α equivalent to the set A (two sets A and B are called **equivalent** if there exists a bijective mapping $\alpha : A \to B$ from the set A onto the set B, see Definition 2.1).

The formal definition of these cardinal numbers and their elementary properties are the main topic of the present unit.

The Theorem of Cantor and Bernstein and the Comparability Theorem (see Sections 2 to 5):

Even if we do not have a formal definition of cardinal numbers, we can compare two sets: Given two sets A and B we say that the sets A and B are **of the same cardinality** if they are equivalent.

The next step in this direction is to compare two sets A and B in the sense that the set A is "smaller" than the set B. For that purpose we will introduce the term that a set B *dominates* the set A:

A set B **dominates a set** A if the set A is equivalent to a subset B' of the set B (see Definition 2.3).
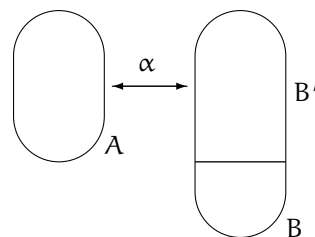
In this case we write $A \preceq B$. If the sets A and B are not equivalent, we write $A \prec B$. This approach and the notation $A \preceq B$ expresses the hope that the relation $\preceq$ behaves more or less as the relation $\leqslant$ and allows to compare arbitrary sets.

In fact, this is the case: The Theorem of Cantor and Bernstein (Theorem 2.11) says that for each two sets A and B the relations

$$A \preceq B \text{ and } B \preceq A$$

imply that the sets A and B are equivalent.

The Comparability Theorem (Theorem 4.4) says that for each two sets A and B we have

$$A \preceq B \text{ or } B \preceq A.$$

Putting these two theorems together we obtain the following important result (Theorem 4.5): For each two sets A and B exactly one of the following relations holds:

(i) We have $A \sim B$.

(ii) We have $A \prec B$.

(iii) We have $B \prec A$.

An important tool for the proof of these results is the following characterization of dominating sets (see Theorem 2.9): Let A and B be two sets. Then the following conditions are equivalent:

(i) The set B dominates the set A, that is, $A \preceq B$.

(ii) There exists an injective mapping $\alpha : A \to B$ from the set A into the set B.

(iii) There exists a surjective mapping $\beta : B \to A$ from the set B onto the set A.

As a consequence the Theorem of Cantor and Bernstein implies the following theorem (see Theorem 2.14):

Let A and B be two sets, and suppose that there exist two injective functions $\alpha : A \to B$ and $\beta : B \to A$. Then there exists a bijective function $\gamma : A \to B$ from the set A onto the set B.
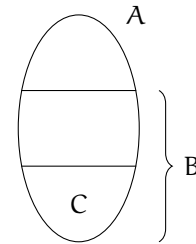
At a first glance the Theorem of Cantor and Bernstein looks quite trivial, in particular, since it can easily been deduced from the following theorem (see Theorem 3.2 and Remark 3.3):

Let A be a set which is equivalent to a subset C of the set A. If B is subset of the set A with

$$C \subseteq B \subseteq A,$$

then the set B is equivalent to the sets A and C.

This is one of the many examples of a theorem which sounds trivial, but whose proof is not at all evident.

We have included different proofs for the Theorem of Cantor and Bernstein and for the Comparability Theorem: Let us start with the Theorem of Cantor and Bernstein. The proof of Julius König explained in Theorem 2.11 is the most direct proof of this theorem. The proof of Dedekind and Zermelo explained in Theorem 3.2 uses the Dedekind chains developed by Richard Dedekind for the formal definition of the natural numbers. Finally, the proof of Felix Bernstein explained in Theorem 3.6 is the first proof of the Theorem of Cantor and Bernstein. It uses cardinal arithmetic that we will explain in Unit *Cardinal Arithmetic* [Garden 2020k].

The first proof of the Comparability Theorem stems from Ernst Zermelo and is explained in Theorem 4.4. In Theorem 5.3 we will explain a second proof based on the Lemma of Zorn.

Cardinal Numbers (see Section 6):

As we have explained above, we are looking for each set A for one distinguished set $\mathfrak{a}$ such that $A \sim \mathfrak{a}$. This set $\mathfrak{a}$ will be defined as the cardinality of the set A and denoted by $|A| := \mathfrak{a}$.

Let us start with a set A. By the well-ordering theorem (see Theorem 4.2), there exists an order $\leqslant_A$ on the set A such that the pair $(A, \leqslant_A)$ is well-ordered. A main theorem about ordinal numbers says that there exists (exactly) one ordinal number $Z = (Z, \leqslant)$ isomorphic to the pair $(A, \leqslant_A)$ (see Theorem 6.4). Since any two ordinal numbers X and Y are comparable (we always have $X \subseteq Y$ or $Y \subseteq X$, see Theorem 6.5), we obtain the existence of the set

$$\mathcal{Z}_A := \{X \subseteq Z \mid X \text{ ordinal number and } X \sim A\}.$$

A further important result about ordinal numbers says that each set of ordinal numbers is well-ordered with respect to the inclusion relation (see Theorem 6.6). It follows that the set $\mathcal{Z}_A$ has a minimal element $\mathfrak{a}$. This element $\mathfrak{a}$ is the distinguished set $\mathfrak{a}$ with $A \sim \mathfrak{a}$ we have been looking for. The above described procedure may be illustrated as follows:

$$A \to (A, \leqslant_A) \to (Z, \leqslant) \to \mathcal{Z}_A \to \mathfrak{a}.$$

This train of thought motivates Definition 6.2 which says that a **cardinal number** $\mathfrak{a}$ is the "smallest" ordinal number Z such that $Z \sim A$ (see also Theorem 8.4). Theorem 6.7 proves that this definition is perfect: For each set A there exists exactly one cardinal number $\mathfrak{a}$ such that $A \sim \mathfrak{a}$.

At the end of this section we will show that given a non-empty set A the sets equivalent to the set A do not form a set (see Theorem 6.14).

The Order of the Cardinal Numbers (see Section 7):

Since cardinal numbers are by definition ordinal numbers and since ordinal numbers are totally ordered by inclusion (we have $A \subseteq B$ or $B \subseteq A$ for all ordinal numbers A and B), the order on the ordinal numbers may be induced to the cardinal numbers: We set

$$\mathfrak{a} \leqslant \mathfrak{b} \text{ if and only if } \mathfrak{a} \subseteq \mathfrak{b} \text{ for all cardinal numbers } \mathfrak{a} \text{ and } \mathfrak{b}.$$

The following result (Theorem 7.5) is the bridge between the dominating sets introduced above and the cardinal numbers: If A and B are two sets, then we have

$$A \preceq B \text{ if and only if } |A| \leqslant |B|.$$

As a consequence we obtain the following characterization (see Corollary 7.6):

Let A and B be two sets. Then the following conditions are equivalent:

(i) We have $|A| \leqslant |B|$.

(ii) There exists an injective mapping $\alpha : A \to B$ from the set A into the set B.

(iii) There exists a surjective mapping $\beta : B \to A$ from the set B onto the set A.

The Theorem of Cantor (see Section 8):

Since the set $\mathbb{N}_0$ of the natural numbers is infinite, we know that there exists at least one infinite cardinal number. It was Georg Cantor who observed in 1874 [Cantor 1874] that there is no bijective mapping between the set of the natural numbers and the set of the real numbers implying that

$$|\mathbb{R}| > |\mathbb{N}_0|.$$

Hence, there exist at least two infinite cardinal numbers. In fact, there are even infinitely many cardinal numbers, a result that follows from the following theorem of Cantor (see Theorem 8.1):

Let A be a set. Then we have

$$|\mathcal{P}(A)| > |A|$$

where $\mathcal{P}(A)$ denotes the power set of the set A, that is, the set of all subsets of the set A. There are even so many cardinal numbers that they do not fit in a set. More formally, the cardinal numbers do not from a set (see Theorem 8.6).

Finite Sets (see Section 9):

In Unit *Finite Sets and their Cardinalities* [Garden 2020h] we have introduced the cardinality of a finite set: If a set A is equivalent to the set n for a natural number n, then the set A is called **finite**, and we write $|A| = n$ (see Definition 9.1).

So we have two definitions of the cardinality of finite sets. Fortunately, the two definitions fit together since every natural number is also a cardinal number (see Theorem 9.5).

At the end of this section we will explain that the set $\mathbb{N}_0$ of the natural numbers is also a cardinal number (see Theorem 9.6).

Infinite Sets (see Section 10):

We will conclude this unit with some observations about infinite cardinal numbers: Obviously, the cardinal number $\mathbb{N}_0$ is infinite (see Proposition 10.1). In fact, the cardinal number $\mathbb{N}_0$ (also called $\aleph_0$, see Definition 9.8) is the smallest infinite cardinal number (see Theorem 10.2). In particular, a set A is infinite if and only if $|A| \geqslant \mathbb{N}_0$.

Infinite sets can also be characterized in a quite different way (see Theorem 10.5): A set A is infinite if and only if there exists a bijective mapping $\alpha : A \to B$ from the set A onto a proper subset B of the set A, that is, if and only if the set A is equivalent to one of its proper subsets.

A set A of cardinality less or equal than $\mathbb{N}_0$ is called **countable** (see Definition 10.6). The term *countable* is due to the fact that there exists a bijective mapping

$$\alpha : A \to \{1, 2, \ldots, n\} \text{ or } \alpha : A \to \mathbb{N}_0 = \{0, 1, 2, \ldots\}$$

from the set A onto the set $\{1, 2, \ldots, n\}$ or onto the set $\mathbb{N}_0$ (or if $A = \emptyset$). In other words, we can count the elements of the set A. Obviously, subsets of countable sets are countable (see Proposition 10.7).

2   The Theorem of Cantor and Bernstein

We will explain what it means that one set B dominates another set A denoted by A $\preceq$ B (see Definition 2.3). A first important result is Theorem 2.9 saying that a set B dominates a set A if and only if there exists an injective mapping $\alpha : A \to B$ from the set A into the set B.

The main result of this section is the Theorem of Cantor and Bernstein saying that the relations A $\preceq$ B and B $\preceq$ A imply that the sets A and B are equivalent.

As a consequence we will see that the existence of two *injective* mappings $\alpha : A \to B$ and $\beta : B \to A$ implies the existence of a *bijective* mapping $\gamma : A \to B$ (see Theorem 2.14).

Equivalent Sets:

Let us recall the definition and basic facts about equivalent sets:

2.1 Definition. Two sets A and B are called **equivalent** if there exists a bijective function $\alpha : A \to B$ from the set A onto the set B. If A and B are two equivalent sets, we write A $\sim$ B.

For more details about functions and equivalent sets see Unit *Functions and Equivalent Sets* [Garden 2020d]. We will need the following result about equivalent sets:

2.2 Proposition. *Let* A, B *and* C *be three sets.*

(a) We have A $\sim$ A *(reflexivity).*

(b) If A $\sim$ B, *then we have* B $\sim$ A *(symmetry).*

(c) If A $\sim$ B *and* B $\sim$ C, *then we have* A $\sim$ C *(transitivity).*

Proof.   See Unit *Functions and Equivalent Sets* [Garden 2020d].   □

Domination of Sets:

2.3 Definition. Let A and B be two sets.

(a) We say that the set B **dominates the set A** if there exists a subset B′ of the set B equivalent to the set A, that is, there exists a bijective function $\alpha : A \to B′$ from the set A onto a subset B′ of the set B.

(b) If the set B dominates the set A, then we write A $\preceq$ B.

(c) If the set B dominates the set A and if the sets A and B are not equivalent, then we say that the set B **strictly dominates the set A**. In this case, we write A $\prec$ B.
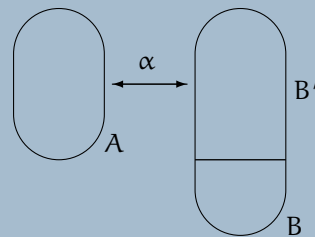
French / German.   Dominating Set = Ensemble dominant = Dominierende Menge.

2.4 Proposition. *Let* A *and* B *be two sets. If the set* A *is a subset of the set* B*, then we have* $A \preceq B$*. In particular, every set dominates the empty set.*

Proof. The assertion follows from the fact that $A \sim A$ (Proposition 2.2) and that the set A is a subset of the set B. □

2.5 Examples. (a) We have $\{4, 5\} \preceq \{1, 2, 3\}$.

(b) We have $\mathbb{N} \preceq \mathbb{N}_0$ and $\mathbb{N}_0 \preceq \mathbb{N}$: The first inequality follows from Proposition 2.4, the second inequality follows from the fact that the mapping $\mathbb{N}_0 \to \{2, 3, 4, \ldots\}$, $n \mapsto n + 2$ is a bijective mapping from the set $\mathbb{N}_0$ onto the subset $\{2, 3, 4, \ldots\}$ of the set $\mathbb{N}$.

In the proof of Theorem 2.9 we will need the following elementary results about embeddings and injective functions:

2.6 Definition. Let B be a set, and let A be a subset of the set B. Then the mapping $\alpha : A \to B$, $\alpha : x \mapsto x$ is called the **embedding** of the set A into the set B.

Note that the embedding $\alpha : A \to B$ is an injective function.

2.7 Proposition. *Let* A*,* B *and* C *be three sets, and let* $\alpha : A \to B$ *and* $\beta : B \to C$ *be two injective functions. Then the function* $\beta \circ \alpha : A \to C$ *is also injective.*

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020d]. □

Finally, we will need the following consequence of the axiom of choice in the proof of Theorem 2.9:

2.8 Theorem. *Let* $\mathcal{C}$ *be a non-empty set of non-empty sets. Then there exists a function*

$$f : \mathcal{C} \to \bigcup_{C \in \mathcal{C}} C$$

such that the element $f(X)$ *is contained in the set* X *for all elements* X *of the set* $\mathcal{C}$*. The function* $f : \mathcal{C} \to \bigcup_{C \in \mathcal{C}} C$ *is called a* **choice function***.*

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020e]. □

2.9 Theorem. *Let* A *and* B *be two non-empty sets. Then the following conditions are equivalent:*

(i) The set B *dominates the set* A*, that is,* $A \preceq B$*.*

(ii) There exists an injective function $\alpha : A \to B$ *from the set* A *into the set* B*.*

(iii) There exists a surjective function $\beta : B \to A$ *from the set* B *onto the set* A*.*
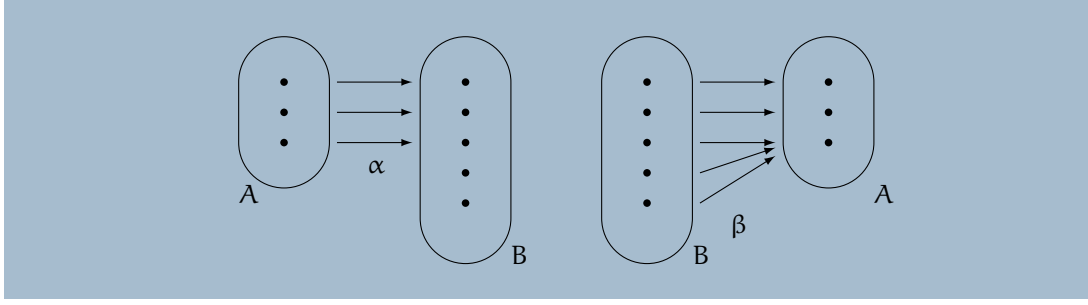
Proof. (i) $\Rightarrow$ (ii): Suppose that the set B dominates the set A. Then there exists a subset B′ of the set B and a bijective function $\beta : A \to B'$ from the set A onto the set B′. Let $\gamma : B' \to B$ be the embedding of the set B′ into the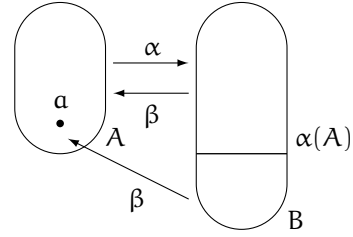 set B (Definition 2.6), and let $\alpha : A \to B$ be defined by $\alpha := \gamma \circ \beta$. By Proposition 2.7, the function $\alpha : A \to B$ is an injective function from the set A into the set B.

(ii) $\Rightarrow$ (iii): Suppose that there exists an injective function $\alpha : A \to B$ from the set A into the set B. Then the function $\alpha : A \to \alpha(A)$ is bijective. Let $\mathfrak{a}$ be an arbitrary element of the set A. Define the function $\beta : B \to A$ from the set B into the set A by

$$\beta(y) := \begin{cases} \alpha^{-1}(y) & \text{if} \quad y \in \alpha(A) \\ \mathfrak{a} & \text{if} \quad y \notin \alpha(A) \end{cases}$$

Since the function $\alpha^{-1} : \alpha(A) \to A$ is bijective, it follows that the function $\beta : B \to A$ is surjective.

(iii) $\Rightarrow$ (i): Suppose that there exists a surjective function $\beta : B \to A$ from the set B onto the set A. It follows that the set $\beta^{-1}(x) := \{y \in B \mid \beta(y) = x\}$ is non-empty for every element x of the set A. By the axiom of choice (see Theorem 2.8), we may choose for each element x of the set A an element b_x of the set $\beta^{-1}(x)$.

Let $B' := \{b_x \mid x \in A\}$. It follows that the function $\alpha : A \to B'$, $x \mapsto b_x$ is bijective. $\square$

The Theorem of Cantor and Bernstein:

We will need the following elementary result about functions in the proof of Theorem 2.11:

2.10 Proposition. *Let $A_1, \ldots, A_n$ be a family of n pairwise disjoint sets, let $B_1, \ldots, B_n$ be a second family of n pairwise disjoint sets, and suppose that there exists for each $j = 1, \ldots, n$ a bijective function $f_j : A_j \to B_j$ from the set A_j onto the set B_j. Let*

$$A := \bigcup_{j=1}^{n} A_j \text{ and } B := \bigcup_{j=1}^{n} B_j.$$

Then there exists a bijective function $f : A \to B$ from the set A onto the set B such that

$$f(x) = f_j(x) \text{ for all } x \in A_j.$$

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020d]. $\square$

We will present the Theorem of Cantor and Bernstein with a proof by Julius König [König 1906]. See also the historical notes at the end of Section 3.

> **2.11 Theorem. (Cantor and Bernstein)** *Let* A *and* B *be two sets.*
>
> *If* $A \preceq B$ *and* $B \preceq A$, *then we have* $A \sim B$.

Proof.   (König) (*) Note that $A \sim B$ implies $A \preceq B$.

Suppose that $A \preceq B$ and $B \preceq A$. The proof goes roughly as follows: By Theorem 2.9, we have two injective functions

$$\alpha : A \to B \text{ and } \beta : B \to A \text{ (Step 2)}.$$

We will split the sets A and B into three disjoint subsets

$$A = X_A \uplus X_B \uplus X_\infty \text{ and } B = Y_A \uplus Y_B \uplus Y_\infty$$

and show that

$$\alpha(X_A) = Y_A, \beta(Y_B) = X_B \text{ and } \alpha(X_\infty) = Y_\infty \text{ (Step 8 to Step 11)}.$$

In Step 12 we will construct a bijective mapping $\gamma : A \to B$ by

$$\gamma := \begin{cases} \alpha & : \ X_A \to Y_A \\ \beta^{-1} & : \ X_B \to Y_B \\ \alpha & : \ X_\infty \to Y_\infty. \end{cases}$$

Step 1. *W.l.o.g. we may suppose that* $A \cap B = \emptyset$:

Let $A' := A \times \{0\}$ and $B' := B \times \{1\}$. Then we have $A \sim A'$ and $B \sim B'$. It follows from $A \preceq B$ and $B \preceq A$ that $A' \preceq B'$ and $B' \preceq A'$. If we can show $A' \sim B'$, then it follows that $A \sim B$.

Step 2. *There exists an injective function* $\alpha : A \to B$ *from the set* A *into the set* B *and an injective function* $\beta : B \to A$ *from the set* B *into the set* A:

The assertions follows from Theorem 2.9.

Step 3. *Definition of a successor and of the notation* $r \to s$:

For an element x of the set A, we call the element $\alpha(x) \in B$ the **successor** of the element x. For an element y of the set B, we call the element $\beta(y) \in A$ the **successor** of the element y.

If r is an element of the set $A \cup B$ and if the element $s \in A \cup B$ is the successor of the element r, then we write $r \to s$. In this case, we call the element r the **predecessor** of the element s. Note that an element x of the set A has a predecessor if and only if the element x is contained in the set $\beta(B)$. Analogously, an element y of the set B has a predecessor if and only if we the element y is contained in the set $\alpha(A)$.

If r and s are two elements of the set $A \cup B$ such that there exist elements $z_0, z_1, \ldots, z_n$ of the set $A \cup B$ such that

$$r = z_0 \to z_1 \to \ldots \to z_n = s,$$

then we say that **there exists a sequence from the element** r **to the element** s.

Step 4. *Let* s *be an element of the set* $A \cup B$. *Then there exists at most one predecessor of the element* s.
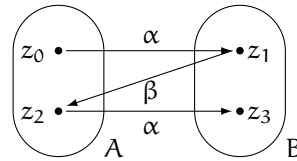
Let r_1 and r_2 be two predecessors of the element s.

If s is an element of the set B, then the elements r_1 and r_2 are both contained in the set A, and we have $\alpha(r_1) = \alpha(r_2) = s$. Since the mapping $\alpha : A \to B$ is injective, it follows that $r_1 = r_2$.

If s is an element of the set A, then the elements r_1 and r_2 are both contained in the set B, and we have $\beta(r_1) = \beta(r_2) = s$. Since the mapping $\beta : B \to A$ is injective, it follows that $r_1 = r_2$.

Step 5. *Definition of a starting point:*

Let r be an element of the set $A \cup B$. The element r is called a **starting point** if the element r does not have a predecessor.

Step 6. *Let s be an element of the set $A \cup B$, and suppose that there are two sequences*

$$
\begin{aligned}
r &= z_n \to z_{m-1} \to \ldots \to z_1 \to z_0 = s \ \textit{and} \\
r' &= z'_m \to z'_{m-1} \to \ldots \to z'_1 \to z'_0 = s.
\end{aligned}
$$

If the elements r and r' are both starting points, then we have $m = n$, $z_j = z'_j$ for all $j = 0, \ldots, m = n$ and $r = r'$:

W.l.o.g. we may suppose that $m \leqslant n$. Since each element of the set $A \cup B$ has at most one predecessor (Step 4), it follows that $z_j = z'_j$ for all $j = 0, \ldots, m$. Since $r' = z'_m$ is a starting point, it follows that $z'_m = z_m = r$ (otherwise, the element z_m is no starting point, in contradiction to $z_m = z'_m = r'$). It follows that $m = n$ and $r = r'$.

Step 7. *Let s be an element of the set $A \cup B$. Then exactly one of the following cases occurs:*

(i) There exists a sequence from a starting point a of the set A to the element s.

(ii) There exists a sequence from a starting point b of the set B to the element s.

(iii) There exists no sequence from a starting point of the set $A \cup B$ to the element s:

Obviously, each element s of the set $A \cup B$ fulfills at least one of the Conditions (i), (ii) or (iii). It is also obvious that Conditions (i) and (iii) and Conditions (ii) and (iii) exclude each other. It remains to show that Conditions (i) and (ii) exclude each other:

Assume on the contrary that there exist two sequences from two starting points a of the set A and b of the set B to the element s. It follows from Step 6 that $a = b \in A \cap B = \emptyset$ (Step 1), a contradiction.

Step 8. *Let*

$$
\begin{aligned}
X_A &:= \{x \in A \mid \textit{there is a sequence from a starting point } a \in A \textit{ to } x\}. \\
X_B &:= \{x \in A \mid \textit{there is a sequence from a starting point } b \in B \textit{ to } x\}. \\
X_\infty &:= \{x \in A \mid \textit{there is no sequence from a starting point } z \in A \cup B \textit{ to } x\}. \\
Y_A &:= \{y \in B \mid \textit{there is a sequence from a starting point } a \in A \textit{ to } y\}. \\
Y_B &:= \{y \in B \mid \textit{there is a sequence from a starting point } b \in B \textit{ to } y\}. \\
Y_\infty &:= \{y \in B \mid \textit{there is no sequence from a starting point } z \in A \cup B \textit{ to } y\}.
\end{aligned}
$$

Then we have $A = X_A \uplus X_B \uplus X_\infty$ and $B = Y_A \uplus Y_B \uplus Y_\infty$:

The assertion follows from Step 7.

Step 9. *We have $\alpha(X_A) = Y_A$:*

Let x be an element of the set X_A. Then there exists a starting point a of the set A and a sequence $a \to \ldots \to x$ from the starting point a to the element x.

Hence, we obtain the sequence $a \to \ldots \to x \to \alpha(x)$ from the starting point a to the element $\alpha(x)$ implying that the element $\alpha(x)$ is contained in the set Y_A. Hence, we have $\alpha(X_A) \subseteq Y_A$.

Conversely, let y be an element of the set Y_A. Then there exists a starting point a of the set A and a sequence $a \to \ldots \to y$ from the starting point a to the element y. Since the element y is contained in the set B and since the element a is contained in the set A, it follows from $A \cap B = \emptyset$ that the element y has a predecessor x in the sequence $a \to \ldots \to x \to y$. It follows that the element x is contained in the set X_A and that $\alpha(x) = y$. Hence, we have $Y_A \subseteq \alpha(X_A)$.

Step 10. *We have $\beta(Y_B) = X_B$:*

The proof is as in Step 9.

Step 11. *We have $\alpha(X_\infty) = Y_\infty$:*

Step 11.1 *We have $\alpha(X_\infty) \subseteq Y_\infty$:*

Let x be an element of the set X_∞. Assume that the element $\alpha(x)$ is not contained in the set Y_∞. By Step 8, the element $\alpha(x)$ is either contained in the set Y_A or in the set Y_B.

If the element $\alpha(x)$ were in the set Y_A or in the Y_B, then the element x would be in the set X_A or X_B, respectively, in contradiction to the fact that $x \in X_\infty$.

Step 11.2 *We have $Y_\infty \subseteq \alpha(X_\infty)$:*

Let y be an element of the set Y_∞. It follows that the element y has a predecessor x of the A. If the element x were in the set X_A or in the X_B, then the element y would be in the set Y_A or Y_B, respectively, in contradiction to the fact that the element y is contained in the set Y_∞. It follows that the element x is contained in the set X_∞. It follows that the element $y = \alpha(x)$ is contained in the set $\alpha(X_\infty)$.

Step 12. *We have $A \sim B$:*

Since the functions $\alpha : A \to B$ and $\beta : B \to A$ are injective, it follows from Step 9 to Step 11 that there exist bijective functions $\gamma_1 : X_A \to Y_A$, $\gamma_2 : X_B \to Y_B$ and $\gamma_3 : X_\infty \to Y_\infty$. Since $A = X_A \cup X_B \cup X_\infty$ and $B = Y_A \cup Y_B \cup Y_\infty$ (Step 8), there exists a bijective function $\gamma : A \to B$ from the set A onto the set B implying that $A \sim B$ (Proposition 2.10). $\qquad\square$

2.12 Remark. The Theorem of Cantor and Bernstein is also called the Theorem of Schröder and Bernstein because the first proofs have been published independently by Ernst Schröder and Felix Bernstein. However, it turned out that there was an error in Schröder's proof. For more details see the historical notes at the end of Section 3.

The Theorem of Cantor and Bernstein is also called the **equivalence theorem**.

French / German.   Equivalence theorem = Théorème d'équivalence = Äquivalenzsatz.

2.13 Corollary. *Let A, B and C be three sets.*

(a) We have $A \preceq A$ (reflexivity).

(b) We have $A \preceq B$ and $B \preceq A$ if and only if $A \sim B$ (symmetry).

(c) If $A \preceq B$ and $B \preceq C$, then we have $A \preceq C$ (transitivity).

Proof. (*) Note that A ∼ B implies A ≼ B.

(a) By Proposition 2.2, we have A ∼ A. It follows from (*) that A ≼ A.

(b) Step 1. *If A ∼ B, then we have A ≼ B and B ≼ A:*

Let A ∼ B. By Proposition 2.2, we have B ∼ A. It follows from (*) that A ≼ B and B ≼ A.

Step 2. *Suppose that A ≼ B and B ≼ A. Then we have A ∼ B:*

The assertion follows from the Theorem of Cantor and Bernstein (Theorem 2.11).

(c) Suppose that we have A ≼ B and B ≼ C for three sets A, B and C. Then there exist two injective functions $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ from the set A into the set B and from the set B into the set C, respectively. It follows from Proposition 2.7 that the function $\gamma : A \rightarrow C$ defined by $\gamma := \beta \circ \alpha$ is injective. Hence, we get A ≼ C (Theorem 2.9). □

> **2.14 Theorem.** *Let A and B be two sets, and suppose that there exist two injective functions $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$. Then there exists a bijective function $\gamma : A \rightarrow B$.*

Proof. The existence of the injective functions $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$ implies that A ≼ B and B ≼ A (Theorem 2.9). It follows from the Theorem of Cantor and Bernstein (Theorem 2.11) that A ∼ B. □

Historical Notes:

See the historical notes at the end of Section 3.

3 Further Proofs of the Theorem of Cantor and Bernstein

We will present two further proofs of the Theorem of Cantor and Bernstein which are due to Richard Dedekind and Ernst Zermelo and to Felix Bernstein.

The Proof of Richard Dedekind and of Ernst Zermelo:

In fact, Dedekind and Zermelo prove a theorem which is equivalent to the Theorem of Cantor and Bernstein, namely Theorem 3.2. Their proof is based on the theory of Dedekind chains developed by Richard Dedekind in his fundamental paper *Was sind und was sollen die Zahlen?* [Dedekind 1888]. We first recall the definition of a Dedekind chain:

> **3.1 Definition.** Let A be a set, and let $\alpha : A \rightarrow A$ be a mapping from the set A into itself.
>
> (a) A subset D of the set A is called a **Dedekind chain with respect to the mapping** $\alpha : A \rightarrow A$ if we have
>
> $$\alpha(D) \subseteq D.$$
>
> If no confusion may arise, we call the set D just a **Dedekind chain** without mentioning the mapping $\alpha : A \rightarrow A$.

> (b) Let B be a subset of the set A. Then the set
>
> $$D(B) \quad := \quad \bigcap \{D \subseteq A \mid B \subseteq D \text{ and } \alpha(D) \subseteq D\}$$
>
> $$= \quad \bigcap \{D \text{ Dedekind chain } \mid B \subseteq D\}$$
>
> is called the **Dedekind chain generated by the set** B.

For more information about Dedekind chains see Unit *The Natural Numbers and the Principle of Induction* [Garden 2020g].

> **3.2 Theorem. (Dedekind and Zermelo)** *Let A be a set which is equivalent to a subset B of the set A. If C is subset of the set A with*
>
> $$B \subseteq C \subseteq A,$$
>
> *then the set C is equivalent to the sets A and B.*

Proof. We will show that the set C is equivalent to the set B. Since the sets A and B are equivalent, it then follows from Proposition 2.2 that the set C is also equivalent to the set A.

Since the sets A and B are equivalent, there exists a bijective mapping $\alpha : A \to B$ from the set A onto the set B.

Step 1. *Definition of the sets Q and $\mathcal{T}$:*

Let $Q := C \setminus B$, and let

$$\mathcal{T} := \{T \subseteq A \mid Q \subseteq T \text{ and } \alpha(T) \subseteq T\}.$$

Step 2. *The set A is contained in the set $\mathcal{T}$. In particular, we have $\mathcal{T} \neq \emptyset$:*

The assertion is obvious.

Step 3. *Definition of the set R:*

Let

$$R := \bigcap_{T \in \mathcal{T}} T$$

be the Dedekind chain generated by the set Q.

Step 4. *The set R is an element of the set $\mathcal{T}$:*

Since the set Q is a subset of all sets T of the set $\mathcal{T}$, we have

$$Q \subseteq \bigcap_{T \in \mathcal{T}} T = R.$$

In order to show that the set $\alpha(R)$ is a subset of the set R, let x be an element of the set R. Since

$$R = \bigcap_{T \in \mathcal{T}} T,$$

the element x is contained in each set T of the set $\mathcal{T}$. By definition of the set $\mathcal{T}$ (Step 1), we have

$$\alpha(x) \in T \text{ for all } T \in \mathcal{T}$$

implying that

$$\alpha(x) \in \bigcap_{T \in \mathcal{T}} T = R.$$

Step 5. *We have $\alpha(R) = R \setminus Q$:*

Step 5.1. *We have*

$$\alpha(R) \subseteq R \setminus Q :$$

Since the set R is an element of the set $\mathcal{T}$, we have

$$\alpha(R) \subseteq R.$$

Since $\alpha : A \to B$ is a mapping from the set A onto the set B and since $Q \cap B = \emptyset$, we have $\alpha(R) \cap Q = \emptyset$. It follows that
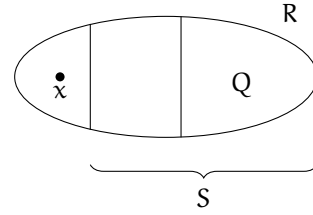
$$\alpha(R) \subseteq R \setminus Q.$$

Step 5.2. *We have*

$$R \setminus Q \subseteq \alpha(R) :$$

Let x be an element of the set $R \setminus Q$. Assume that the element x is not contained in the set $\alpha(R)$.

Let $S := R \setminus \{x\}$. Since the element x is not contained in the set Q and since the set Q is a subset of the set R, the set Q is also a subset of the set S.

We claim that the set $\alpha(S)$ is a subset of the set S: For, let s be an element of the set $S = R \setminus \{x\}$. Since the set R is contained in the set $\mathcal{T}$ (Step 4), the element $\alpha(s)$ is contained in the set R. Since, by assumption, the element x is not contained in the set $\alpha(R)$, we have $\alpha(s) \neq x$. Altogether, we get

$$\alpha(s) \in R \setminus \{x\} = S.$$

Hence, the set $\alpha(S)$ is a subset of the set S.

Since the set Q and the set $\alpha(S)$ are subsets of the set S, the set S is contained in the set $\mathcal{T}$. It follows that

$$S = R \setminus \{x\} \subset R = \bigcap_{T \in \mathcal{T}} T \subseteq S,$$

a contradiction. Hence, the element x is contained in the set $\alpha(R)$ implying that

$$R \setminus Q \subseteq \alpha(R).$$

Step 6. *We have $R = Q \uplus \alpha(R)$:*

Since the set R is an element of the set $\mathcal{T}$, the set Q is a subset of the set R. It follows from Step 5 that

$$R = (R \setminus Q) \uplus Q = \alpha(R) \uplus Q.$$

Step 7. *The sets* B *and* C *are equivalent:*

Let $B' := B \setminus \alpha(R)$. It follows that $B = B' \uplus \alpha(R)$ implying that

$$
\begin{aligned}
C &= B \uplus Q = \big(B' \uplus \alpha(R)\big) \uplus Q \\
&= B' \uplus \big(\alpha(R) \uplus Q\big) = B' \uplus R.
\end{aligned}
$$

Since the mapping $\alpha : A \to B$ is bijective, the mapping $\alpha|_R : R \to \alpha(R)$ is also bijective. Obviously, the mapping $\mathrm{id} : B' \to B'$, $x \mapsto x$ is bijective.
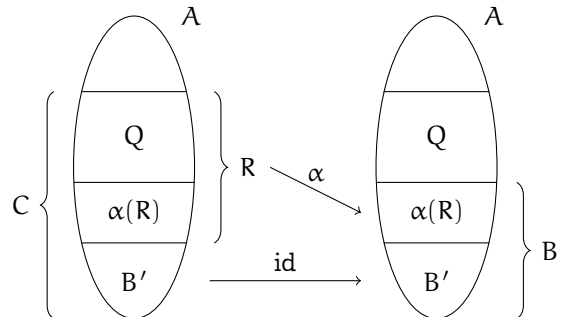
Since $C = R \uplus B'$ and $B = \alpha(R) \uplus B'$, it follows from Proposition 2.10 that there exists a bijective mapping from the set C onto the set B, that is, the sets B and C are equivalent.   □

3.3 Remark. We present an alternative proof of the Theorem of Cantor and Bernstein (Theorem 2.11) based on the Theorem of Dedekind and Zermelo (Theorem 3.2). More precisely, we will show that the Theorem of Cantor and Bernstein and the Theorem of Dedekind and Zermelo are easy consequences of each other.

Proof.   Step 1. *The Theorem of Dedekind and Zermelo is an easy consequence of the Theorem of Cantor and Bernstein:*

Let A, B and C be three sets such that

$$B \subseteq C \subseteq A,$$

and suppose that the sets A and B are equivalent. By Proposition 2.4, we have $B \preceq C$ and $C \preceq A$. Since $A \sim B$, it follows from Corollary 2.13 that $A \preceq C$. By the Theorem of Cantor and Bernstein, we get $A \sim C$.

Step 2. *The Theorem of Cantor and Bernstein is an easy consequence of the Theorem of Dedekind and Zermelo:*

Let A and B be two sets with $A \preceq B$ and $B \preceq A$. Then there exist a subset A' of the set A, a subset B' of the set B and two bijective mappings $\alpha : A \to B'$ and $\beta : B \to A'$ from the sets A and B onto the sets B' and A', respectively.

Set $\gamma := \beta \circ \alpha : A \to A$, and let

$$A'' := \gamma(A) = \beta(\alpha(A)) = \beta(B') \subseteq \beta(B) = A'.$$
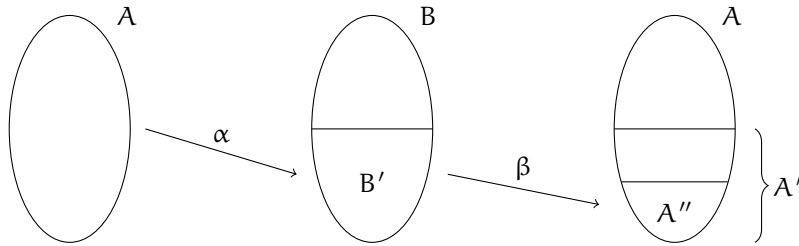
It follows that $\gamma : A \to A''$ is a bijective mapping, that is, $A \sim A''$. Since we have

$$A'' \subseteq A' \subseteq A,$$

it follows from Theorem 3.2 that $A \sim A'$. Since $B \sim A'$, it follows from Proposition 2.2 that $A \sim B$.   □

The Proof of Felix Bernstein:

We will need the following results for the proof of Bernstein:

3.4 Proposition. *Let A be a finite set, and let B be a subset of the set A. Then the set*

B *is finite, and there exist two natural numbers* m *and* n *such that*

$$A \sim n, \ B \sim m \ \text{and} \ m \leqslant n.$$

Proof. See Unit *Finite Sets and their Cardinalities* [Garden 2020h]. □

3.5 Theorem. *Let* A *and* B *be two sets.*

(a) Suppose that the set A *is finite and that the set* B *is infinite. Then the set* $A \cup B$ *is equivalent to the set* B.

(b) Suppose that the sets A *and* B *are both infinite. If the sets* A *and* B *are equivalent, then the set* A *is also equivalent to the set* $A \cup B$.

Proof. We will present the proof of this theorem in Unit *Cardinal Arithmetic* [Garden 2020k]. □

3.6 Theorem. (Cantor and Bernstein) *Let* A *and* B *be two sets with* $A \preceq B$ *and* $B \preceq A$. *Then we have* $A \sim B$.

Proof. (Bernstein) Step 1. *Definition of the sets* A_1 *and* B_1 *and of the mappings* $\alpha : A \to B_1$ *and* $\beta : B \to A_1$:

Let A and B be two sets with $A \preceq B$ and $B \preceq A$. It follows that there exists a subset A_1 of the set A and a subset B_1 of the set B such that $A_1 \sim B$ and $B_1 \sim A$. In particular, there exist two bijective mappings $\alpha : A \to B_1$ and $\beta : B \to A_1$ from the sets A and B onto the sets B_1 and A_1, respectively.

Step 2. *We may suppose that the set* A_1 *is a proper subset of the set* A *and that the set* B_1 *is a proper subset of the set* B:

Otherwise, it follows immediately that $A \sim B$.

Step 3. *We may suppose that the two sets* A *and* B *are infinite:*

W.l.o.g. suppose that the set A is finite. It follows from Proposition 3.4 that the set A_1 is finite and that there exist two natural numbers m and n such that

$$A \sim n, \ A_1 \sim m \ \text{and} \ m \leqslant n.$$

Since $B \sim A_1$ and $B_1 \sim A$, we have $B \sim m$ and $B_1 \sim n$. Since the set B_1 is a subset of the set B, it follows from Proposition 3.4 that $n \leqslant m$. Hence, we have $m = n$ and

$$A \sim n \sim B.$$

Step 4. *Let*
$$B_2 := \alpha(A_1), \ A_2 := \beta(B_1) \ \text{and} \ A_3 := \beta(B_2).$$

Then the set A_3 *is infinite, and we have*

$$A_3 \subset A_2 \subset A_1 \subset A \ \text{and} \ B_2 \subset B_1 \subset B:$$

Since the mappings $\alpha : A \to B$ and $\beta : B \to A$ are injective and since $\alpha(A_1) = B_2$ and $\beta(B_2) = A_3$, we have
$$B \sim A_1 \sim B_2 \sim A_3.$$

Since the set B is infinite (Step 3), the set A_3 is also infinite.

Since the set B_1 is a proper subset of the set B and since the mapping $\beta : B \to A$ is injective, the set $A_2 = \beta(B_1)$ is a proper subset of the set $A_1 = \beta(B)$.

Since the set A_1 is a proper subset of the set A and since the mapping $\alpha : A \to B$ is injective, the set $B_2 = \alpha(A_1)$ is a proper subset of the set $B_1 = \alpha(A)$.

Since the set B_2 is a proper subset of the set B_1 and since the mapping $\beta : B \to A$ is injective, the set $A_3 = \beta(B_2)$ is a proper subset of the set $A_2 = \beta(B_1)$.

Step 5. *We have $(A \setminus A_1) \sim (A_2 \setminus A_3)$:*

Since the mappings $\alpha : A \to B$ and $\beta : B \to A$ are injective, the mapping $\gamma := \beta \circ \alpha : A \to A$ is injective. Since

$$\begin{aligned}
\gamma(A) &= (\beta \circ \alpha)(A) = \beta(\alpha(A)) = \beta(B_1) = A_2 \text{ and} \\
\gamma(A_1) &= (\beta \circ \alpha)(A_1) = \beta(\alpha(A_1)) = \beta(B_2) = A_3,
\end{aligned}$$

it follows that

$$\gamma(A \setminus A_1) = A_2 \setminus A_3$$

implying that $(A \setminus A_1) \sim (A_2 \setminus A_3)$.

Step 6. *We have $A \sim B$:*

We have

$$\begin{aligned}
A &= (A \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3 \text{ and} \\
A_1 &= \phantom{(A \setminus A_1) \cup {}} (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3.
\end{aligned}$$

Case 1. *Suppose that the set $A \setminus A_1$ is finite.*

Since the set A_3 is infinite (Step 4), it follows from Theorem 3.5 that the set $(A \setminus A_1) \cup A_3$ is equivalent to the set A_3. It follows that the set $A = (A \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3$ is equivalent to the set $A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3$. Since $A_1 \sim B$, we get $A \sim B$.

Case 2. *Suppose that the set $A \setminus A_1$ is infinite.*

Since $(A \setminus A_1) \sim (A_2 \setminus A_3)$ (Step 5), it follows from Theorem 3.5 that the set $(A \setminus A_1) \cup (A_2 \setminus A_3)$ is equivalent to the set $(A_2 \setminus A_3)$. It follows that the set $A = (A \setminus A_1) \cup (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3$ is equivalent to the set $A_1 = (A_1 \setminus A_2) \cup (A_2 \setminus A_3) \cup A_3$. Since $A_1 \sim B$, we get $A \sim B$.   □

Historical Notes:

The Theorem of Cantor and Bernstein has first been announced by Georg Cantor:

Hat man irgendeine wohldefinierte Menge M der zweiten Mächtigkeit, eine Teilmenge M′ von M und eine Teillmenge M″ von M′ und weiß man, dass die letztere M″ gegenseitig eindeutig abbildbar ist auf die erste M, so ist immer auch die zweite M′ gegenseitig eindeutig abbildbar auf die erste und daher auf die dritte.

[...] es scheint mir aber höchst bemerkenswert und hebe ich es daher ausdrücklich hervor, dass dieser Satz allgemeine Gültigkeit hat, gleichviel, welche Mächtigkeit der Menge M zukommen mag. Darauf will ich einer späteren Abhandlung näher eingehen [...].

See [Cantor 1883, pp. 582 - 583].

If one has some well-defined set M *of the second cardinality, a subset* M′ *of* M *and a subset* M″ *of* M′ *and one knows that the latter* M″ *can be clearly mapped bijectively onto the first* M, *then the second* M′ *can always be mapped bijectively onto the first and therefore bijectively onto the third.*

[...] but it seems to me to be extremely remarkable and I therefore emphasize it expressly that this proposition has general validity, regardless of the cardinality of the set M. *I will go into this in more detail in a later treatise [...].*

A set of the second cardinality means that the set is equivalent to the set $\mathbb{R}$ of the real numbers. So Cantor proved Theorem 3.2 for the special case that M ∼ $\mathbb{R}$ and announced a proof for the general case of Theorem 3.2, but he never published a proof. As we have seen in Remark 3.3, the Theorem of Cantor and Bernstein and Theorem 3.2 follow easily from each other.

The first proofs of the Theorem of Cantor and Bernstein have been published by Ernst Schröder [Schröder 1898] and by Felix Bernstein. Unfortunately, the proof of Schröder was flawed as has been pointed out by Alwin Reinhold Korselt [Korselt 1911]. The proof of Felix Bernstein has been published by Emile Borel [Borel 1898]. We have explained the proof of Bernstein in Theorem 3.6.

Lorsqu'on est dans l'un des derniers cas, peut-on affirmer que les deux ensembles A *et* B *ont même puissance ?*

Nous allons démontrer qu'il en est ainsi dans le troisième cas ; mais dans le quatrième cas nous ne savons rien.

See [Borel 1898, p. 103].

When we are in one of the last cases, can we say that the two sets A *and* B *have the same cardinality?*

We will show that this is the case in the third case; but in the fourth case we know nothing.

The last two cases Borel mentions are as follows:

3. *Il existe un* A₁ *[⊆* A*] ayant même puissance que* B *et aussi un* B₁ *[⊆* B*] ayant même puissance que* A.

4. *Il n'existe ni un* A₁ *[⊆* A*] ayant même puissance que* B *ni un* B₁ *[⊆* B*] ayant même puissance que* A.

See [Borel 1898, p. 102].

3. *There exists an* A₁ *[⊆* A*] with the same cardinality as* B *and also a* B₁ *[⊆* B*] with the same cardinality as* A.

4. *There does not exist neither an* A₁ *[⊆* A*] with the same cardinality as* B *nor a* B₁ *[⊆* B*] with the same cardinality as* A.

So Case 3 is exactly the case B $\preceq$ A and A $\preceq$ B. In a footnote Borel explains that this theorem has first been announced by Cantor and then proved by Bernstein and that he presents the proof of Bernstein.

[...] cette démonstration inédite est due a M. Felix Bernstein et a été donnée pour la première fois dans le séminaire de M. G. Cantor, à Halle.

See [Borel 1898, p. 142] (footnote (3)).

[...] this so far unpublished demonstration is due to Mr. Felix Bernstein and was given for the first time in the seminar of Mr. G. Cantor, in Halle.

For the fourth case see the historical notes at the end of Section 5.

Bernstein visited Richard Dedekind and told him about his result. Thereupon Dedekind drafted his own proof for this theorem, based on his theory of chains (for more details about the Dedekind chains see Unit *The Natural Numbers and the Principle of Induction* [Garden 2020g]) without publishing it. This proof was found in Dedekind's estate by Jean Cavaillès.[1]

In einem Brief vom 29. August 1899 schreibt Dedekind über den Satz an Cantor: "Als der junge Herr Felix Bernstein mich Pfingsten 1897 in Harzburg besuchte, sprach er von dem Satz [...] und stutzte ein wenig, als ich meine Überzeugung aussprach, dass derselbe mit meinen Mitteln (Was sind und was sollen die Zahlen?) leicht zu beweisen sei [...]. Nach seiner Abreise setzte ich mich daran und konstruierte den hier beiliegenden Beweis." [...]

Den ursprünglichen Beweis fand J. Cavaillès - Paris im Nachlass.

See [Dedekind 1932, p. 448].

In a letter dated August 29, 1899, Dedekind wrote about the theorem to Cantor: "When the young Mr. Felix Bernstein visited me at Whitsun 1897 in Harzburg, he spoke of the theorem [...] and was a little taken aback when I expressed my conviction that the same is easy to prove with my means (Was sind und was sollen die Zahlen?) [...]. After his departure, I sat down and constructed the proof enclosed here. [...]

J. Cavaillès - Paris found the original proof in the estate.

It is Emmy Noether, the editor of the Complete Works of Richard Dedekind, who explains this history. She included the manuscript of Dedekind with the proof of 1887 in the Collected Works of Dedekind [Dedekind 1932]. In this manuscript the Theorem of Cantor and Bernstein reads as follows:

Ist S ähnlich in sich selbst abgebildet, ist also das Bild $\varphi(S) = S' \subseteq S$, ist ferner $S' \subseteq T \subseteq S$, so ist auch T dem S ähnlich.

See [Dedekind 1932, p. 447].

If S is mapped into itself by a similarity, that is, the image $\varphi(S)$ fulfills $\varphi(S) = S' \subseteq S$ and if $S' \subseteq T \subseteq S$, then T and S are similar.

A similarity is an injective mapping, similar means equivalent. So Dedekind's theorem is in fact Theorem 3.2. The proof of Dedekind is more or less the proof of Theorem 3.2 presented above.

This proof has been found independently from Dedekind by Ernst Zermelo who published it in [Zermelo 1908, p. 272] (Theorem 27). Before that Henri Poincaré [Poincaré 1906, p. 314] published the result of Zermelo, but expressed his doubts about the correctness of the proof of Zermelo.

If you want to study the manuscript of Dedekind, the remark of Emmy Noether is quite helpful that Dedekind included a complete proof with exception of one proposition, namely Proposition 63 of [Dedekind 1888]. The proof of this proposition is left in [Dedekind 1888] as an exercise, but explained by Zermelo. It corresponds to Step 5 of the proof of Theorem 3.2.

Finally, we want to point out that the proof of the Theorem of Cantor and Bernstein explained in Theorem 2.11 is due to Julius König [König 1906].

---

[1] Jean Cavaillès and Emmy Noether were the editors of the letters exchanged between Georg Cantor and Richard Dedekind [Cavaillès, Jean and Noether, Emmy 1937].

4 The Comparability Theorem

The main result of this section is the Comparability Theorem saying that we have $A \preceq B$ or $B \preceq A$ for each two sets A and B (see Theorem 4.4).

We will use the following fundamental results about well-ordered sets in the proof of Theorem 4.4. We first recall the definition of well-ordered sets.

4.1 Definition. An ordered set $A = (A, \leqslant)$ is called a **well-ordered set** if any non-empty subset of the set A has a minimal element.

For more details see Unit *Well-Ordered Sets* [Garden 2020i].

4.2 Theorem. (Well-Ordering Theorem) *Let A be a set. Then there exists an order $\leqslant$ on the set A such that the pair $(A, \leqslant)$ is a well-ordered set.*

Proof. See Unit *Well-Ordered Sets* [Garden 2020i]. □

4.3 Theorem. *Let $A(A, \leqslant_A)$ and $B = (B, \leqslant_B)$ be two well-ordered sets. Then exactly one of the following cases occurs:*

(i) We have $(A, \leqslant_A) \cong (B, \leqslant_B)$.

(ii) There exists an element a of the set A such that the set B is isomorphic to the initial segment $A_a := \{x \in A \mid x <_A a\}$ of the set A.

(iii) There exists an element b of the set B such that the set A is isomorphic to the initial segment $B_b := \{y \in B \mid y <_B b\}$ of the set B.

Proof. See Unit *Well-Ordered Sets* [Garden 2020i]. □

4.4 Theorem. (Comparability Theorem) *Let A and B be two sets. Then we have $A \preceq B$ or $B \preceq A$.*

Proof. Let A and B be two sets. By Theorem 4.2, there exist well-orderings $\leqslant_A$ and $\leqslant_B$ such that the pairs $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are well-ordered sets. By Theorem 4.3, one of the following cases occurs:

(i) We have $(A, \leqslant_A) \cong (B, \leqslant_B)$ implying that $A \sim B$.

(ii) There exists an element a of the set A such that the set B is isomorphic to the initial segment $A_a := \{x \in A \mid x <_A a\}$ of the set A. It follows that $B \sim A_a$ implying that $B \preceq A$.

(iii) There exists an element b of the set B such that the set A is isomorphic to the initial segment $B_b := \{y \in B \mid y <_B b\}$ of the set B. It follows that $A \sim B_b$ implying that $A \preceq B$.

It follows from (i) to (iii) that $A \sim B$ or $A \prec B$ or $B \prec A$. □

French / German. Comparability Theorem = Théorème de Comparaison = Vergleichbarkeitssatz.

We summarize the results of the Theorem of Cantor and Bernstein (Theorem 2.11 and Corollary 2.13) and of the Comparability Theorem (Theorem 4.4) in the following theorem:

> **4.5 Theorem.** *Let* A *and* B *be two sets. Then exactly one of the following cases occurs:*
>
> *(i) We have* A $\sim$ B*.*
>
> *(ii) We have* A $\prec$ B*.*
>
> *(iii) We have* B $\prec$ A*.*

Proof.   It follows from Theorem 4.4 that A $\preceq$ B or B $\preceq$ A. Hence, we have A $\sim$ B or A $\prec$ B or B $\prec$ A.

Obviously, the cases A $\sim$ B and A $\prec$ B and the cases A $\sim$ B and B $\prec$ A exclude each other. Assume that A $\prec$ B and B $\prec$ A. Then we have A $\preceq$ B and B $\preceq$ A, and it follows from Theorem 2.11 that A $\sim$ B, in contradiction to A $\prec$ B. $\square$

> **4.6 Proposition.** *Let* A, B *and* C *be three sets.*
>
> *(a) If* A $\sim$ B*, then we have* A $\preceq$ B *and* B $\preceq$ A
>
> *(b) If* A $\preceq$ B *and* B $\prec$ C*, then we have* A $\prec$ C*.*
>
> *(c) If* A $\prec$ B *and* B $\preceq$ C*, then we have* A $\prec$ C*.*
>
> *(d) If* A $\sim$ B *and* B $\preceq$ C*, then we have* A $\preceq$ C*.*
>
> *(e) If* A $\preceq$ B *and* B $\sim$ C*, then we have* A $\preceq$ C*.*
>
> *(f) If* A $\sim$ B *and* B $\prec$ C*, then we have* A $\prec$ C*.*
>
> *(g) If* A $\prec$ B *and* B $\sim$ C*, then we have* A $\prec$ C*.*

Proof.   (a) is obvious.

(b) It follows from Corollary 2.13 that A $\preceq$ C. Assume that A $\sim$ C. Then we have C $\preceq$ A. It follows from C $\preceq$ A and A $\preceq$ B that C $\preceq$ B, in contradiction to the assumption that B $\prec$ C (Theorem 4.4).

(c) follows as (b).

(d) follows from (a) and Corollary 2.13.

(e) follows from (a) and Corollary 2.13.

(f) follows from (a) and (b).

(g) follows from (a) and (c). $\square$

Historical Notes:

See the historical notes at the end of Section 5.

5   A Second Proof of the Comparability Theorem

We will present an alternative proof of the Comparability Theorem (Theorem 4.4) based on the Lemma of Zorn (see Theorem 5.3). We first recall the Lemma of Zorn. We will need Proposition 5.2 about functions in the proof of Theorem 5.3.

5.1 Theorem. (Lemma of Zorn) *Let $A = (A, \leqslant)$ be a partially ordered set such that every chain C of the set A has an upper bound in the set A. (A chain of the set A is a totally ordered subset of the set A.)*

Then the set A contains a maximal element.

Proof. See Section *Ordered Sets and the Lemma of Zorn* [Garden 2020f]. □

5.2 Proposition. *Let I be a totally ordered index set, and let $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be two families of sets with the following properties:*

(i) The families $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ are chains, that is, we have

$$A_i \subseteq A_j \text{ and } B_i \subseteq B_j \text{ if } i \leqslant j.$$

(ii) For each element i of the set I, there exists a function $\alpha_i : A_i \to B_i$ from the set A_i into the set B_i.

(iii) For each two elements i and j of the set I such that $i \leqslant j$, the function $\alpha_i : A_i \to B_i$ is induced by the function $\alpha_j : A_j \to B_j$, that is, we have

$$\alpha_i(x) = \alpha_j(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i \in I} A_i \text{ and } B := \bigcup_{i \in I} B_i.$$

(a) There exists exactly one function $\alpha : A \to B$ from the set $A = \bigcup_{i \in I} A_i$ into the set $B = \bigcup_{i \in I} B_i$ such that $\alpha|_{A_i} = \alpha_i$ for all elements i of the set I.

(b) If the functions $\alpha_i : A_i \to B_i$ are bijective for all elements i of the set I, then the function $\alpha : A \to B$ is also bijective.

Proof. For a proof see Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020f]. □

5.3 Theorem. (Comparability Theorem) *Let A and B be two sets. Then we have $A \preceq B$ or $B \preceq A$.*

Proof. Let A and B be two sets. We want to show that $A \preceq B$ or $B \preceq A$:

Let

$$\mathcal{F} := \{f : A' \to B' \mid A' \subseteq A, B' \subseteq B \text{ and } f : A' \to B' \text{ bijective}\}$$

be the set of the bijective functions from a subset A' of the set A onto a subset B' of the set B.

We define the following partial order on the set $\mathcal{F}$: For two functions $f : A_1 \to B_1$ and $g : A_2 \to B_2$ set $f \leqslant g$ if and only if the set A_1 is a subset of the set A_2, the set B_1 is a subset of the set B_2 and if we have

$$f_2(x) = f_1(x) \text{ for all } x \in A_1.$$

We claim that the set $\mathcal{F}$ has a maximal element: For, let $I = (I, \leqslant)$ be a totally ordered index set, and let $(f_i)_{i \in I}$ be a chain of functions of the set $\mathcal{F}$, that is, of bijective functions $f_i : A_i \to B_i$ with

$$A_i \subseteq A_j \subseteq A \text{ and } B_i \subseteq B_j \subseteq B \text{ for all } i, j \in I \text{ with } i \leqslant j$$

and

$$f_i(x) = f_j(x) \text{ for all } x \in A_i \text{ and all } i, j \in I \text{ with } i \leqslant j.$$

Let

$$A' := \bigcup_{i \in I} A_i \text{ and } B' := \bigcup_{i \in I} B_i.$$

By Proposition 5.2, there exists a bijective function $f : A' \to B'$ from the set A' onto the set B' with the property

$$f(x) = f_i(x) \text{ for all } i \in I.$$

Since the sets A' and B' are subsets of the sets A and B, respectively, the function $f : A' \to B'$ is an upper bound of the chain $(f_i)_{i \in I}$.

By the Lemma of Zorn (Theorem 5.1), there exists a maximal element $g : A'' \to B''$ of the set $\mathcal{F}$ for two subsets A'' and B'' of the sets A and B, respectively.

Case 1. Suppose that $A'' = A$. In this case we have $g : A \to B''$, and the subset B'' of the set B is equivalent to the set A, that is, $A \preceq B$.

Case 2. Suppose that $B'' = B$. In this case we have $g : A'' \to B$, and the subset A'' of the set A is equivalent to the set B, that is, $B \preceq A$.

Case 3. Suppose that $A'' \subset A$ and $B'' \subset B$. Then there exist two elements

$$a \in A \setminus A'' \text{ and } b \in B \setminus B''.$$

Define the function $h : A'' \cup \{a\} \to B'' \cup \{b\}$ by

$$h(x) := \begin{cases} g(x) & \text{if } x \in A'' \\ b & \text{if } x = a \end{cases}$$

Obviously, the function $h : A'' \cup \{a\} \to B'' \cup \{b\}$ is an element of the set $\mathcal{F}$ with $g < h$, in contradiction to the fact that the function $g : A'' \to B''$ is a maximal element of the set $\mathcal{F}$.

It follows from Case 1 to 3 that $A \preceq B$ or $B \preceq A$.   □

Historical Notes:

Note that we will introduce the cardinality $|A|$ of a set A in Section 6 (Definition 6.2), and that we will show in Theorem 6.10 that we have

$$A \sim B \text{ if and only if } |A| = |B|.$$

The Comparability Theorem (Theorem 4.4) has been formulated for the first time by Georg Cantor:

Sind die beiden Mannigfaltigkeiten M *und* N *nicht von gleicher Mächtigkeit, so wird entweder* M *mit einem Bestandteil von* N*, oder es wird* N *mit einem Bestandteil von* M *gleiche Mächtigkeit haben.*

See [Cantor 1878, p. 242].

If the two sets M *and* N *are not of the same cardinality, then either* M *has the same cardinality as a subset of* N *or* N *has the same cardinality as a subset of* M*.*

Cantor did not give a proof for this theorem. The theorem has been proven by Zermelo in his famous paper [Zermelo 1904] where he proves that every set can be well-ordered:

[…] und je zwei Mengen sind miteinander "vergleichbar", d.h. es ist immer die eine ein-eindeutig abbildbar auf die andere oder einen ihrer Teile.

[Zermelo 1904, p. 516]

[…] and any two sets are "comparable", that is, one set may always be mapped bijectively onto the other set or onto a subset of the other set.

The proof of Zermelo is more or less the proof of Theorem 4.4 explained above.

Note that Felix Bernstein proved the following weaker version of the Comparability Theorem in his PhD-thesis [Bernstein 1901] (see also [Bernstein 1905, p. 131]):

If $|M| + |N| = |M| \cdot |N|$, then the sets M and N are comparable.

Note that we will explain the meaning of $|M| + |N|$ and of $|M| \cdot |N|$ in Unit *Cardinal Arithmetic* [Garden 2020k].

By the way, the comparability theorem answers Case (iv) of Borel mentioned in the historical notes at the end of Section 3.

6   Cardinal Numbers

We finally come to the main topic of this unit, namely the definition of the cardinal numbers. It will be explained in Definition 6.2. The main theorem is Theorem 6.7 saying that for each set A there exists exactly one cardinal number $\mathfrak{a}$ such that $A \sim \mathfrak{a}$.

At the end of this section we will explain that the sets equivalent to a given set $A \neq \emptyset$ do not form a set (see Theorem 6.14).

Cardinal numbers are specific ordinal numbers. We therefore first recall the definition of ordinal numbers:

> **6.1 Definition.** Let $A = (A, \leqslant)$ be an ordered set. The set A is called an **ordinal number** if the pair $A = (A, \leqslant)$ is well-ordered and if we have
>
> $$\mathfrak{a} = A_{\mathfrak{a}} = \{x \in A \mid x < \mathfrak{a}\} \text{ for all } \mathfrak{a} \in A.$$
>
> In other words each element $\mathfrak{a}$ of the set A equals the initial segment with respect to the element $\mathfrak{a}$.

For more details about ordinal numbers see Unit *Ordinal Numbers* [Garden 2020j].

> **6.2 Definition.** Let A be an ordinal number. The ordinal number A is called a **cardinal number** if it fulfills the following condition: For each ordinal number B satisfying
>
> $$B \subseteq A \text{ and } B \sim A$$
>
> we have $B = A$.

French / German.   Cardinal number = Nombre cardinal = Kardinalzahl.

> **6.3 Remark.** Let A be a set. In Theorem 8.2 we will show that the set
>
> $$\mathcal{A} := \{X \text{ ordinal number } \mid X \sim A\}$$

exists. We will see in Theorem 8.4 that

$$|A| = \min\{X \text{ ordinal number } \mid X \sim A\}.$$

A main result about cardinal numbers is Theorem 6.7 saying that for each set A there exists exactly one cardinal number $\mathfrak{a}$ such that $\mathfrak{a} \sim A$. For the proof of this theorem we need the following results about well-ordered sets and about ordinal numbers:

6.4 Theorem. *Let* $(A, \leqslant_A)$ *be a well-ordered set. Then there exists exactly one ordinal number* $N = (N, \leqslant_N)$ *isomorphic to the set* A.

Proof.   See Unit *Ordinal Numbers* [Garden 2020j].   □

6.5 Theorem. *Let* $(A, \leqslant_A)$ *and* $B = (B, \leqslant_B)$ *be two ordinal numbers.*

(a) We have
$$A \subseteq B \text{ or } B \subseteq A.$$

(b) We have
$$A \subset B \text{ if and only if } A \in B.$$

Proof.   See Unit *Ordinal Numbers* [Garden 2020j].   □

6.6 Theorem. *Let* A *be a set of ordinal numbers. Then the set* $A = (A, \subseteq)$ *is well-ordered. In particular, it has a minimal element.*

Proof.   See Unit *Ordinal Numbers* [Garden 2020j].   □

6.7 Theorem. *Let* A *be a set. Then there exists exactly one cardinal number* $\mathfrak{a}$ *such that* $A \sim \mathfrak{a}$.

Proof.   Let A be a set.

Step 1. *Existence of the cardinal number* $\mathfrak{a}$:

By Theorem 4.2, there exists an order $\leqslant_A$ on the set A such that the pair $(A, \leqslant_A)$ is well-ordered. By Theorem 6.4, there exists an ordinal number Z such that the pair $(Z, \subseteq)$ is isomorphic to the pair $(A, \leqslant_A)$. In particular, we have $A \sim Z$. Let

$$\mathcal{Z} := \{X \subseteq Z \mid X \text{ is an ordinal number and } X \sim A\}.$$

By Theorem 6.6, the set $\mathcal{Z}$ is well-ordered and has therefore a minimal element $\mathfrak{a}$. We claim that the ordinal number $\mathfrak{a}$ is a cardinal number:

For, let b be an ordinal number satisfying

$$b \subseteq \mathfrak{a} \text{ and } b \sim \mathfrak{a}.$$

By Theorem 6.5, we have

$$b \subseteq Z \text{ or } Z \subseteq b.$$

If the ordinal number b is a proper subset of the set Z, then it follows from Theorem 6.5 that the ordinal number b is an element of the set $\mathcal{Z}$. Since the ordinal number $\mathfrak{a}$ is the minimum of the set $\mathcal{Z}$, we get

$$\mathfrak{a} \subseteq b$$

implying $b = a$.

If the ordinal number Z is a subset of the set b, we get

$$a \subseteq Z \subseteq b$$

implying $b = a$.

Step 2. *Uniqueness of the cardinal number* a:

Let b be a second cardinal number with $b \sim A$. By Theorem 6.5, we have

$$a \subseteq b \text{ or } b \subseteq a.$$

Since both, a and b are cardinal numbers, it follows in both cases that $a = b$.   □

6.8 Definition. Let A be a set, and let a be the unique cardinal number with $A \sim a$ (Theorem 6.7). The cardinal number a is called the **cardinality of the set** A. It is denoted by $|A| := a$.

French / German.   Cardinality of a set = Cardinalité = Kardinalität or Mächtigkeit.

6.9 Remark. We will see in Theorem 9.5 that every natural number n is a cardinal number. Therefore, Definition 6.2 is an extension of the definition of the cardinality of a finite set introduced in Unit *Finite Sets and their Cardinalities* [Garden 2020h].

6.10 Theorem. *Let A and B be two sets. Then the following conditions are equivalent:*

(i) The sets A and B are equivalent.

(ii) We have $|A| = |B|$.

Proof.   The assertion follows from Theorem 6.7.   □

6.11 Corollary. *Let A be a cardinal number. Then we have $|A| = A$.*

Proof.   The assertion follows from Theorem 6.10 since A and $|A|$ are two cardinal numbers with $A \sim |A|$.   □

In Theorem 6.14 we will show that the sets equivalent to a given non-empty set A do not form a set. For the proof of Theorem 6.14 we recall the definition of an ordered pair and of the direct product of two sets. In addition, we will need the axiom of foundation.

6.12 Remark. Let a and b be two elements of two sets A and B, respectively.

(a) The ordered pair (a, b) is defined by $(a, b) := \{a, \{a, b\}\}$.

(b) The direct product $A \times B$ of the sets A and B is defined by

$$A \times B := \{(x, y) \mid x \in A \text{ and } y \in B\}.$$

For more details about direct products see Unit *Direct Products and Relations* [Garden 2020c].

6.13 Axiom. (Axiom of Foundation) Every non-empty set Z contains an element R which is element minimal with respect to the set Z, that is, an element R such that

$$R \cap Z = \emptyset.$$

For more details about the axiom of foundation see Unit *Unions and Intersections of Sets* [Garden 2020b].

6.14 Theorem. *Let A be a non-empty set. There does not exist a set $\mathcal{A}$ such that*

$$X \in \mathcal{A} \text{ if and only if } X \sim A.$$

In other words, the sets equivalent to the set A do not form a set.

Proof.   Assume that there exists a set $\mathcal{A}$ such that

$$\mathcal{A} = \{X \mid X \sim A\}.$$

Since the direct product $\{\mathcal{A}\} \times A$ is equivalent to the set A, the set $\{\mathcal{A}\} \times A$ is an element of the set $\mathcal{A}$, that is, there exists an element B of the set $\mathcal{A}$ such that

$$\{\mathcal{A}\} \times A = B.$$

Since $A \neq \emptyset$, there exists an element $\mathfrak{a}$ of the set A. Set

$$R_0 := (\mathcal{A}, \mathfrak{a}), \ R_1 := \{\mathcal{A}\} \times A \text{ and } R_2 := \mathcal{A}.$$

By Remark 6.12, we have

$$\begin{aligned} R_0 &= (\mathcal{A}, \mathfrak{a}) \in \{\mathcal{A}\} \times A = R_1, \\ R_1 &= \{\mathcal{A}\} \times A = B \in \mathcal{A} = R_2 \text{ and} \\ R_2 &= \mathcal{A} \in \{\mathcal{A}, \{\mathcal{A}, \mathfrak{a}\}\} = (\mathcal{A}, \mathfrak{a}) = R_0. \end{aligned}$$

Let $Z := \{R_0, R_1, R_2\}$. Then we have

$$R_0 \in R_1 \cap Z, \ R_1 \in R_2 \cap Z \text{ and } R_2 \in R_0 \cap Z,$$

in contradiction to the axiom of foundation (Axiom 6.13) saying that there exists an element R of the set Z such that $R \cap Z = \emptyset$. $\qquad \square$

Historical Notes:

The investigation of the cardinality of a set is as old as the process of counting. Natural numbers have always been used to count the number of the elements of a (finite) set. The formal definition of the set $\mathbb{N}_0$ of the natural numbers has been given by Richard Dedekind [Dedekind 1888] in his fundamental paper *Was sind und was sollen die Zahlen?* (for more details see Unit *The Natural Numbers and the Principle of Induction* [Garden 2020g]).

For a long time one was not aware of the fact that there are different magnitudes of infinity or of different infinite cardinal numbers as we would say today. It was Georg Cantor who

discovered this important fact. More precisely, he showed in [Cantor 1874] that the set of the real numbers is of greater cardinality than the set of the natural numbers:

Wenn eine nach irgend einem Gesetz gegebene unendliche Reihe von einander verschie-dener reeller Zahlgrößen:

$$(4.)\ \omega_1, \omega_2, \ldots, \omega_\nu, \ldots$$

vorliegt, so lässt sich in jedem vorgegebenen Intervall $(\alpha \ldots \beta)$ eine Zahl η (und folglich unendlich vieler solcher Zahlen) bestimmen, welche in der Reihe (4.) nicht vorkommt.

See [Cantor 1874, p. 260].

If an infinite series of different real numbers

$$(4.)\ \omega_1, \omega_2, \ldots, \omega_\nu, \ldots$$

is given, a number η (and consequently an infinite number of such numbers) can be determined in every given interval $[\alpha, \beta]$ which does not appear in the series (4.).

The terminology of cardinal numbers has not been developed at that time, so Cantor says that there does not exist a bijective mapping between the set of the natural numbers and the set of the real numbers. Cantor announced this result to Richard Dedekind in a letter of December, 7 in 1873. That's why this date is often called the birthday of set theory (see [Deiser 2020, p. 134]).

The article [Cantor 1874] is the first in a series of six articles entitled *Über unendliche lineare Punktmannigfaltigkeiten*. In the third article of this series [Cantor 1882] Cantor investigates the cardinality of general sets:

[...] so dass zwei Mengen dann aber auch nur dann **gleiche Mächtigleit** *zugestanden wird, wenn sie nach irgendeinem Gesetze einander gegenseitig eindeutig zugeordnet wer-den können.*

See [Cantor 1882, p. 116].

[...] such that two sets are called to be **of the same cardinality** *if one set can be mapped bijectively onto the other set.*

Cantor does not define the cardinality of a set, but he defines when two sets are of the same cardinality. Nevertheless, he also speaks of the cardinality of a set, for example of the cardinality of the set of the positive rational numbers:

[...] die Mächtigkeit der positiven ganzen rationalen Zahlenreihe;

See [Cantor 1882, p. 116].

[...] the cardinality of the set of all positive rational numbers;

Cantor provides the definition of the cardinality of a set in 1895:

Jeder Menge kommt eine bestimmte **Mächtigkeit** *zu, welche wir auch ihre* **Cardinalzahl** *nennen.*

Mächtigkeit *oder* **Cardinalzahl** *von M nennen wir den Allgemeinbegriff, welcher mit Hilfe unseres aktiven Denkvermögens dadurch aus der Menge M hervorgeht, dass von der Beschaffenheit ihrer verschiedenen Elemente $\mathfrak{m}$ und von der Ordnung ihres Gegebenseins abstrahiert wird.*

See [Cantor 1895, p. 481].

Each set has a certain **cardinality** *which we also call its* **cardinal number**.

We call the **cardinality** *or the* **cardinal number** *of* M *the general term which, with the help of our active thinking ability, emerges from the set* M *by abstracting from the nature of its various elements* m *and from the order of their existence.*

This definition is a bit vague. One idea for a more concrete definition is to say that the cardinality of a set A is the set of all sets equivalent to the set A. But we have seen in Theorem 6.14 that the sets equivalent to a set $A \neq \emptyset$ do not form a set.

After the introduction of the axiomatic of Zermelo and Fraenkel John von Neumann [von Neumann 1928] was able to give a precise definition of the cardinality of a set and of cardinal numbers within this set-theoretical framework. His definition is the definition explained in Definition 6.2 and in Definition 6.8.

Definition. *Wir nennen eine Ordnungszahl* P *eine* **Kardinalzahl** *(oder Anfangszahl, Aleph, Mächtigkeit) [...], wenn für kein* Q *OZ [Ordinalzahl],* Q < P, Q ~ P *ist.*

See [von Neumann 1928, p. 731].

Definition. *We call an ordinal number* P *a* **cardinal number** *(or initial number, aleph, cardinality) [...] if we do not have* Q < P *and* Q ~ P *for all ordinal numbers* Q.

Theorem 6.7 and Theorem 6.10 are also contained in [von Neumann 1928]:

Satz 64a. *Sei* H *ein Bereich. Es gibt eine und nur eine Kardinalzahl* P, *für die* P ~ H *ist. Dieses* P *nennen wir die Kardinalzahl (oder Mächtigkeit oder Aleph oder Anfangszahl) von* H *[...].*

Satz 63. *Wenn* P *KZ,* Q *KZ ist, so ist* P = Q *mit* P ~ Q *gleichbedeutend.*

See [von Neumann 1928, pp. 731 - 732].

Theorem 64a. *Let* H *be a domain. There is one and only one cardinal number* P *with* P ~ H. *This* P *is called the cardinal number (or cardinality or aleph or initial number) of* H *[...].*

Theorem 63. *If* P *and* Q *are cardinal numbers, then* P = Q *is equivalent to* P ~ Q.

7   The Order of the Cardinal Numbers

The cardinal numbers can be ordered by the following definition (see Definition 7.1)

$$\mathfrak{a} \leqslant \mathfrak{b} \text{ if and only if } \mathfrak{a} \subseteq \mathfrak{b} \text{ for all cardinal numbers } \mathfrak{a} \text{ and } \mathfrak{b}.$$

The main results of this section are that this order is a total order on any set of cardinal numbers (see Theorem 7.3) and that we have $A \preceq B$ if and only if $|A| \leqslant |B|$ (see Theorem 7.5). Recall that cardinal numbers are by definition ordinal numbers (Definition 6.2). Hence it follows from Theorem 6.5 that we have

$$\mathfrak{a} \subseteq \mathfrak{b} \text{ or } \mathfrak{b} \subseteq \mathfrak{a} \text{ for all cardinal numbers } \mathfrak{a} \text{ and } \mathfrak{b}.$$

7.1 Definition. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers. We set

(a) $\mathfrak{a} \leqslant \mathfrak{b}$ if and only if $\mathfrak{a} \subseteq \mathfrak{b}$.

(b) $\mathfrak{a} < \mathfrak{b}$ if and only if $\mathfrak{a} \subset \mathfrak{b}$.

(c) $\mathfrak{a} \geqslant \mathfrak{b}$ if and only if $\mathfrak{b} \subseteq \mathfrak{a}$.

(d) $\mathfrak{a} > \mathfrak{b}$ if and only if $\mathfrak{b} \subset \mathfrak{a}$.

The so-defined relation $\leqslant$ on each set of cardinal numbers is called the **standard order on the cardinal numbers**.

French / German.   Standard order on the cardinal numbers = Ordre standard sur le nombres ordineaux = Standardordnung auf den Kardinalzahlen.

7.2 Remarks. (a) Since the cardinal numbers do not form a set (see Theorem 6.14), the relation $\leqslant$ is only an order on every set of cardinal numbers and not on the set of all cardinal numbers (which does not exist).

(b) The order on the cardinal numbers is induced by the order on the ordinal numbers.

7.3 Theorem. *Let* A, B *and* C *be three sets, and let* $\leqslant$ *be the standard order of the cardinal numbers.*

(a) We have $|A| \leqslant |A|$.

(b) If $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$, *then we have* $|A| = |B|$.

(c) If $|A| \leqslant |B|$ *and* $|B| \leqslant |C|$, *then we have* $|A| \leqslant |C|$.

(d) Let A *and* B *be two sets. Then we either have* $|A| = |B|$ *or* $|A| < |B|$ *or* $|A| > |B|$.

Proof.   The assertion follows directly from the definition of the order $\leqslant$.   □

7.4 Proposition. *Let* A, B *and* C *be three sets.*

(a) If $|A| = |B|$, *then we have* $|A| \leqslant |B|$ *and* $|B| \leqslant |A|$.

(b) If $|A| \leqslant |B|$ *and* $|B| < |C|$, *then we have* $|A| < |C|$.

(c) If $|A| < |B|$ *and* $|B| \leqslant |C|$, *then we have* $|A| < |C|$.

(d) If $|A| = |B|$ *and* $|B| \leqslant |C|$, *then we have* $|A| \leqslant |C|$.

(e) If $|A| \leqslant |B|$ *and* $|B| = |C|$, *then we have* $|A| \leqslant |C|$.

(f) If $|A| = |B|$ *and* $|B| < |C|$, *then we have* $|A| < |C|$.

(g) If $|A| < |B|$ *and* $|B| = |C|$, *then we have* $|A| < |C|$.

Proof.   The assertion is obvious.   □

7.5 Theorem. *Let* A *and* B *be two sets.*

(a) The following conditions are equivalent:

(i) We have $A \preceq B$.

(ii) We have $|A| \leqslant |B|$.

(iii) We have $|A| \subseteq |B|$.

(b) The following conditions are equivalent:

(i) We have $A \prec B$.

(ii) We have $|A| < |B|$.

(iii) We have $|A| \subset |B|$.

(iv) We have $|A| \in |B|$.

Proof. (a) (i) $\Rightarrow$ (ii): Suppose that $A \preceq B$. Then there exists a subset B' of the set B such that $A \sim B'$. It follows that

$$|A| = |B'| \leqslant |B|.$$

(ii) $\Leftrightarrow$ (iii) follows from Definition 7.1.

(iii) $\Rightarrow$ (i): Suppose that $|A| \subseteq |B|$. Assume that $B \prec A$. It follows from (i) $\Rightarrow$ (iii) that $|B| \subseteq |A|$ implying that $|A| = |B|$. By Theorem 6.10, we have $A \sim B$, in contradiction to $B \prec A$.

(b) (i) $\Rightarrow$ (ii): Suppose that $A \prec B$. By (a), we have $|A| \subseteq |B|$. The assumption $|A| = |B|$ would imply $A \sim B$, in contradiction to $A \prec B$.

(ii) $\Leftrightarrow$ (iii) follows from Definition 7.1.

(iii) $\Leftrightarrow$ (iv): Since $|A|$ and $|B|$ are ordinal numbers, it follows from Theorem 6.5 that (iii) $\Leftrightarrow$ (iv).

(iii) $\Rightarrow$ (i): Suppose that we have $|A| \subset |B|$. Assume that $B \preceq A$. It follows from (a) that $|B| \subseteq |A|$, a contradiction. $\square$

> **7.6 Corollary.** *Let A and B be two sets. The following conditions are equivalent:*
>
> *(i) We have $|A| \leqslant |B|$.*
>
> *(ii) There exists an injective mapping $\alpha : A \to B$ from the set A into the set B.*
>
> *(iii) There exists a surjective mapping $\beta : B \to A$ from the set B onto the set A.*

Proof. The assertion follows from Theorem 2.9 and Theorem 7.5. $\square$

> **7.7 Corollary.** *(a) Let A and B be two sets. If the set A is a subset of the set B, then we have $|A| \leqslant |B|$.*
>
> *(b) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers with $\mathfrak{a} \leqslant \mathfrak{b}$. Then there exist two sets A and B such that*
>
> $$|A| = \mathfrak{a}, \ |B| = \mathfrak{b} \ and \ A \subseteq B.$$

Proof. (a) Let A be a subset of the set B. It follows from Proposition 2.4 that $A \preceq B$. By Theorem 7.5, we have $|A| \leqslant |B|$.

(b) Let $A := \mathfrak{a}$ and $B := \mathfrak{b}$. Obviously, we have $|A| = \mathfrak{a}$ and $|B| = \mathfrak{b}$. By Definition 7.1, it follows from $\mathfrak{a} \leqslant \mathfrak{b}$ that

$$A = \mathfrak{a} \subseteq \mathfrak{b} = B.$$

$\square$

8   The Theorem of Cantor

The main result of this section is the following theorem of Georg Cantor saying that for each set A there exists a set of greater cardinality, namely the power set $\mathcal{P}(A)$ of the set A (see Theorem 8.1).

As a corollary we will see that neither the cardinal numbers nor the ordinal numbers form a set (see Theorem 8.6).

8.1 Theorem. (Cantor) *Let A be an arbitrary set. Then we have*

$$|A| < |\mathcal{P}(A)|$$

where $\mathcal{P}(A)$ denotes the power set of the set A.

Proof.   We claim that $A \prec \mathcal{P}A)$:

Since the function $A \to \mathcal{P}(A)$, $x \mapsto \{x\}$ is an injective function from the set A into the set $\mathcal{P}(A)$, we have $A \preceq \mathcal{P}(A)$ (Theorem 2.9).

Assume that $\mathcal{P}(A) \preceq A$. It follows from Theorem 2.9 that there exists a surjective function $\alpha : A \to \mathcal{P}(A)$ from the set A onto the power set $\mathcal{P}(A)$ of the set A. Let

$$Z := \{x \in A \mid x \notin \alpha(x)\}.$$

Since the set Z is contained in the power set $\mathcal{P}(A)$, there exists an element a of the set A such that $\alpha(a) = Z$.

Case 1. *Suppose that the element a is contained in the set Z. Then we have*

$$a \notin \alpha(a) = Z, \text{ a contradiction.}$$

Case 2. *Suppose that the element a is not contained in the set Z. Then we have*

$$a \in \alpha(a) = Z, \text{ a contradiction.}$$

$\square$

8.2 Theorem. *Let A be a set. Then the following set exists:*

$$\mathcal{A} := \{X \text{ ordinal number } \mid X \sim A\}.$$

Proof.   We want to apply the axiom of specification saying that for all sets X the set

$$Y := \{x \in X \mid \varphi(x)\}$$

exists where φ is some mathematical sentence. See Unit *The Mathematical Universe* [Garden 2020a] for more details.

Let $P := \mathcal{P}(A)$ be the power set of the set A. By Theorem 4.2, there exists an order $\leqslant'$ on the set P such that the pair $(P, \leqslant')$ is well-ordered. By Theorem 6.4, there exists (exactly one) ordinal number C isomorphic to the pair $(P, \leqslant')$.

Step 1. *Let D be an ordinal number equivalent to the set A. Then the set D is a subset of the set C:*

Since the sets C and D are ordinal numbers, it follows from Theorem 6.5 that we have

$$C \subseteq D \text{ or } D \subseteq C.$$

Assume that the set C is a subset of the set D. By Proposition 2.4, we have $C \preceq D$. It follows from $C \sim P$, $C \preceq D$ and $A \sim D$ that $P \preceq A$ (Proposition 4.6).

On the other hand, it follows from Theorem 8.1 that $A \prec \mathcal{P}(A) = P$, a contradiction.

Step 2. *The set $\mathcal{A} := \{X \text{ ordinal number } \mid X \sim A\}$ exists:*

It follows from Step 1 that

$$\mathcal{A} = \{X \in \mathcal{P}(C) \mid X \text{ is an ordinal number and } X \sim A\}.$$

The above set exists in view of the axiom of specification.

$\square$

8.3 Remark. As we have seen in Theorem 6.14, given a non-empty set A there is no set of the form

$$\{X \mid X \sim A\}.$$

In other words, it is necessary in Theorem 8.2 to restrict the set $\mathcal{A}$ to the *ordinal numbers* equivalent to the set A.

8.4 Theorem. *Let A be a set. Then we have*

$$|A| = \min\{X \text{ ordinal number} \mid X \sim A\}.$$

Proof. In view of Theorem 8.2, the set

$$\mathcal{A} := \{X \text{ ordinal number} \mid X \sim A\}$$

exists. By Theorem 6.6, the set $\mathcal{A} = (\mathcal{A}, \subseteq)$ is well-ordered. By definition of a well-ordered set (Definition 4.1), the set $\mathcal{A}$ contains a minimal element M.

In order to show that the minimum M is unique, assume that there is a second minimum M'. It follows that

$$M \subseteq M' \text{ and } M' \subseteq M$$

implying that $M = M'$. It follows from the definition of the cardinality of a set (see Definition 6.2 and Definition 6.8) that $M = |A|$. □

In Theorem 8.2 we have seen that for each set A the set

$$\mathcal{A} := \{X \text{ ordinal number} \mid X \sim A\}$$

exists. However, as we will see in Theorem 8.6, neither the totality of the cardinal numbers nor the totality of the ordinal numbers forms a set. For the proof of this theorem we need the following result about ordinal numbers:

8.5 Theorem. *Let $\mathcal{A}$ be a set of ordinal numbers, and let*

$$S := \bigcup_{X \in \mathcal{A}} X.$$

Then the set S is an ordinal number.

Proof. See Unit *Ordinal Numbers* [Garden 2020j]. □

8.6 Theorem. *(a) There does not exist a set $\mathcal{C}$ with the following property:*

$$X \in \mathcal{C} \text{ if and only if } X \text{ is a cardinal number}$$

in other words, the cardinal numbers do not form a set.

(b) There does not exist a set $\mathcal{C}$ with the following property:

$$X \in \mathcal{C} \text{ if and only if } X \text{ is a ordinal number}$$

Proof.   (a) Assume that there exists a set $\mathcal{C}$ with the following property:

$$X \in \mathcal{C} \text{ if and only if } X \text{ is a cardinal number.}$$

Since each element of the set $\mathcal{C}$ is a cardinal number and therefore an ordinal number, it follows from Theorem 8.5 that the set

$$S := \bigcup_{X \in \mathcal{C}} X$$

is an ordinal number. Since each cardinal number is an element of the set $\mathcal{C}$, we have

$$\mathfrak{a} \subseteq S \text{ for all cardinal numbers } \mathfrak{a}.$$

By Theorem 7.5, it follows that

$$\mathfrak{a} \leqslant |S| \text{ for all cardinal numbers } \mathfrak{a}.$$

On the other hand, it follows from the Theorem of Cantor (Theorem 8.1) that we have

$$|\mathcal{P}(S)| > |S|,$$

a contradiction.

(b) Assume that there exists the set $\mathcal{A}$ of the ordinal numbers. Since every cardinal number is an ordinal number, it follows from the axiom of specification that the set

$$\mathcal{C} := \{A \in \mathcal{A} \mid A \text{ cardinal number}\}$$

exists and consists of all cardinal numbers, in contradiction to (a).   □

Historical Notes:

In the historical notes at the end of Section 6 we have explained that Georg Cantor [Cantor 1874] has shown in 1874 that the cardinality of the set of the real numbers is strictly greater than the cardinality of the set of the natural numbers.

The Theorem of Cantor (Theorem 8.1) extends this result significantly. It has been proven by Georg Cantor in 1890:

[...] dass jeder gegebenen Mannigfaltigkeit L *eine andere* M *an die Seite gestellt werden kann, welche von stärkerer Mächtigkeit ist als* L.

See [Cantor 1890, p. 77].

[...] that there exists for each set L *a further set* M *which is of greater cardinality than* L.

In his proof Cantor uses the set

$$\mathcal{F} := \{f : L \to \{0, 1\} \mid f \text{ is a function}\}.$$

The set $\mathcal{F}$ is equivalent to the power set $\mathcal{P}(L)$ of the set L via the bijective mapping $\mathcal{F} \to \mathcal{P}(L)$, $f \mapsto \{x \in L \mid f(x) = 1\}$. For the rest the proof is identical with the proof of Theorem 8.1 explained above.

Note that Theorem 8.6 (b) is the Theorem of Burali-Forti explained in Unit *Ordinal Numbers* [Garden 2020j].

9   Finite Sets

The main results of this section are Theorems 9.5 and 9.6 saying that each natural number and that the set $\mathbb{N}_0$ of the natural numbers is a cardinal number.

Finite Sets:

Let us first recall some elementary facts about the cardinality of finite sets:

9.1 Definition. (a) A set A is called **finite** if it is equivalent to a natural number n, that is, if there exists a natural number n such that $A \sim n$.

In this case we say that **the set A has n elements**, and we write $|A| = n$.

(b) If a set A is not finite, then it is called **infinite**. In this case we write $|A| = \infty$.

For more details see Unit *Finite Sets and their Cardinalities* [Garden 2020h].

For a finite set A we have two definitions of the cardinality $|A|$ of the set A, namely Definition 6.2 and Definition 9.1. In Theorem 9.5 we will show that each natural number is a cardinal number implying that the two definitions fit together.

9.2 Proposition. *Let m and n be two equivalent natural numbers. Then we have $m = n$.*

Proof.   See Unit *Finite Sets and their Cardinalities* [Garden 2020h].   □

9.3 Proposition. *Let n be a natural number. Then we have $n \prec n + 1$ and $n \prec \mathbb{N}_0$.*

Proof.   Since the set n is a subset of the set $n + 1$, we have $n \preceq n + 1$. By (a), the sets n and $n + 1$ are not equivalent implying that $n \prec n + 1$.

It follows from

$$n \prec n + 1 \preceq \mathbb{N}_0$$

and Proposition 4.6 that $n \prec \mathbb{N}_0$.   □

Natural Numbers and $\mathbb{N}_0$ are Cardinal Numbers:

In Theorem 9.5 and Theorem 9.6 we will show that the natural numbers and that the set $\mathbb{N}_0$ of the natural numbers are cardinal numbers. We will use the following result about ordinal numbers in the the proofs of Theorem 9.5 and of Theorem 9.6.

9.4 Theorem. *(a) The set $\mathbb{N}_0$ of the natural numbers is an ordinal number.*

(b) Let A be an ordinal number. Then exactly one of the following possibilities occurs:

(i) There is a natural number n such that $A = n$.

(ii) We have $\mathbb{N}_0 \leqslant A$.

Proof.   See Unit *Ordinal Numbers* [Garden 2020j].   □

9.5 Theorem. *Every natural number is a cardinal number.*

Proof.   Let n be a natural number, and let

$$A_n := \{x \text{ ordinal number } \mid x \sim n\}.$$

Note that the set A_n exists in view of Theorem 8.2.

Step 1. *Let x be an element of the set A_n. Then the element x is a natural number:*

Since the element x and the set $\mathbb{N}_0$ are ordinal numbers, it follows from Theorem 6.5 that we have

$$x \subset \mathbb{N}_0 \text{ or } \mathbb{N}_0 \subseteq x.$$

Assuming $\mathbb{N}_0 \subseteq x$ it follows from $x \sim n$ that $\mathbb{N}_0 \preceq n$, in contradiction to Proposition 9.3.

By Theorem 6.5, it follows from $x \subset \mathbb{N}_0$ that $x \in \mathbb{N}_0$, that is, the element x is a natural number.

Step 2. *We have $A_n = \{n\}$:*

Let x be an element of the set A_n. By Step 1, the element x is a natural number. It follows from Proposition 9.2 that $x = n$.

Step 3. *The natural number n is a cardinal number:*

By Step 2, we have

$$A_n = \{x \text{ ordinal number } \mid x \sim n\} = \{n\}.$$

It follows that the natural number n is the minimal element of the set A_n (with respect to the standard order of ordinal numbers). Hence, the natural number n is a cardinal number.   □

9.6 Theorem. *The set $\mathbb{N}_0$ of the natural numbers is a cardinal number.*

Proof.   Let

$$A := \{x \text{ ordinal number } \mid x \sim \mathbb{N}_0\}.$$

Since every natural number n fulfills the inequality $n \prec \mathbb{N}_0$ (Proposition 9.3), it follows from Theorem 9.4 that we have

$$\mathbb{N}_0 \leqslant x \text{ for all } x \in A \text{ (standard order of the ordinal numbers)}.$$

Hence the set $\mathbb{N}_0$ is the minimal element of the set A with respect to the standard order of ordinal numbers implying that the set $\mathbb{N}_0$ is a cardinal number.   □

9.7 Remark. The use of $|A| = \infty$ is ambiguous: On the one hand, $|A|$ denotes the cardinality of the set A, on the other hand, the equation $|A| = \infty$ expresses that the set A is infinite. The symbol ∞ is no cardinal number. With this explanation in mind, there should not arise any confusion.

By Theorem 9.4, the cardinal number $\mathbb{N}_0$ is the smallest infinite cardinal number.

9.8 Definition. The cardinal number $\mathbb{N}_0$ is denoted by $\aleph_0$ (speak: Alef zero). ($\aleph$ is the first letter of the hebrew alphabet.)

Historical Notes:

As we have seen in the historical notes of Section 6, John von Neumann has given a formal definition of cardinal numbers in [von Neumann 1928]. In this paper he has also shown that that the natural numbers and that $\aleph_0$ are cardinal numbers (see Theorem 9.5 and Theorem 9.6).

Satz 89. *Aus* P *NZ folgt* P *KZ. [...]*

See [von Neumann 1928, p. 745].

Theorem 89. *It follows from* P *NZ that* P *KZ. [...]*

In other word, if P is a natural number, then P is a cardinal number.

Satz 86. *Es ist* ω *LZ und* ω *KZ. [...]*

See [von Neumann 1928, p. 744].

Theorem 86. *We have* ω *LZ and* ω *KZ. [...]*

In other words, the set ω $(= \mathbb{N}_0 = \aleph_0)$ is a limit number and a cardinal number.

10 Infinite Sets

The main result of this section is Theorem 10.2 saying that the set $\mathbb{N}_0$ is the smallest infinite cardinal number.

Infinite Sets:

10.1 Proposition. *The set $\mathbb{N}_0$ is infinite.*

Proof. Assume that the set $\mathbb{N}_0$ is finite. Then there exists a natural number n such that $\mathbb{N}_0 \sim n$, in contradiction to $n \prec \mathbb{N}_0$ (Proposition 9.3). □

The following theorem says that the ordinal number $\mathbb{N}_0$ is the smallest infinite cardinal number.

10.2 Theorem. *For each set A, the following conditions are equivalent:*

(i) The set A is infinite.

(ii) We have $\mathbb{N}_0 \preceq A$.

(iii) We have $\mathbb{N}_0 = |\mathbb{N}_0| \leqslant |A|$.

Proof. (i) $\Rightarrow$ (iii): Suppose that the set A is infinite. It follows from Theorem 9.4 that $\mathbb{N}_0 \leqslant |A|$.

(iii) $\rightarrow$ (ii) follows from Theorem 7.5.

(ii) $\Rightarrow$ (i): Suppose that the set A fulfills the condition $\mathbb{N}_0 \preceq A$. Assume that the set A is finite. Then there exists a natural number n such that $A \sim n$. It follows from Proposition 10.1 that

$$n \prec \mathbb{N}_0 \preceq A \sim n,$$

in contradiction to Proposition 4.6. □

We will apply Theorem 10.2 to show that every infinite set contains a set of n elements for each natural number n:

> **10.3 Theorem.** *Let A be an infinite set, and let n be a natural number. Then there exists a subset A_n of the set A with exactly n elements.*

Proof.   By Theorem 10.2, we have $\mathbb{N}_0 \leqslant |A|$. It follows from Corollary 7.6 that there exists an injective mapping $\alpha : \mathbb{N}_0 \to A$ from the set $\mathbb{N}_0$ into the set A. Hence, the set

$$A_n := \{\alpha(1), \ldots, \alpha(n)\}$$

is a subset of the set A with exactly n elements. $\qquad\qquad\square$

> **10.4 Proposition.** *For each set A, the following two conditions are equivalent:*
>
> *(i) The set A is finite.*
> *(ii) We have $A \prec \mathbb{N}_0$.*
> *(iii) We have $|A| < |\mathbb{N}_0| = \mathbb{N}_0$.*

Proof.   The assertion follows from Theorem 10.2. $\qquad\qquad\square$

> **10.5 Theorem.** *Let A be a set. Then the following two conditions are equivalent:*
>
> *(i) The set A is infinite.*
> *(ii) There exists a proper subset B of the set A and a bijective function $\alpha : A \to B$ from the set A onto the set B.*

Proof.   (i) $\Rightarrow$ (ii): Suppose that the set A is infinite. By Theorem 10.2, we have $\mathbb{N}_0 \preceq A$, that is, there exists an injective function $\beta : \mathbb{N}_0 \to A$ (Theorem 2.9).

It follows that
$$A = \{\beta(n) \mid n \in \mathbb{N}_0\} \cup C \text{ where } C := A \setminus \{\beta(n) \mid n \in \mathbb{N}_0\}$$

and that the set
$$B := A \setminus \{\beta(0)\} = \{\beta(n) \mid n \in \mathbb{N}\} \cup C$$

is a proper subset of the set A.

Since the mappings $\{\beta(n) \mid n \in \mathbb{N}_0\} \to \{\beta(n) \mid n \in \mathbb{N}\}$, $\beta(n) \mapsto \beta(n+1)$ and $C \to C$, $x \mapsto x$ are bijective, the mapping $\alpha : A \to B$ defined by

$$\alpha(x) := \begin{cases} \beta(n+1) & \text{if} \quad x = \beta(n) \text{ for a number } n \in \mathbb{N}_0 \\ x & \text{if} \quad x \notin \beta(\mathbb{N}_0). \end{cases}$$

is a bijective mapping from the set A onto the set B.

(ii) $\Rightarrow$ (i): Suppose that the set A is finite. Then there exists a natural number n such that $|A| = n$. We prove by induction on n that the set A is not equivalent to a proper subset of the set A.

$n = 0$: The assertion is obvious since the set $A = \emptyset$ has no proper subset.

$n \mapsto n + 1$: Assume that there exists a set A such that $|A| = n + 1$ and a bijective mapping $\alpha : A \to B$ from the set A onto a proper subset B of the set A. Let a be an element of the set A, and let $b := \alpha(a)$. Then the mapping $\alpha : A \to B$ induces a bijective mapping $\alpha' : A \setminus \{a\} \to B \setminus \{b\}$ from the set $A \setminus \{a\}$ onto its proper subset $B \setminus \{b\}$. Since $|A \setminus \{a\}| = n$, this yields a contradiction to the induction hypothesis. $\qquad\qquad\square$

Countable Sets:

10.6 Definition. A set A is called **countable** if we have $A \preceq \mathbb{N}_0$, and it is called **countably infinite** if we have $A \sim \mathbb{N}_0$.

French / German. Countable set = Ensemble dénombrable = Abzählbare Menge.

10.7 Proposition. *(a) Every subset of the set $\mathbb{N}_0$ is countable.*

(b) Every subset of a countable set is countable.

Proof. The assertion follows from Proposition 2.4. □

10.8 Proposition. *Let A be a countably infinite set, and let B be an arbitrary set. Then the following two conditions are equivalent:*

(i) The set B is countable.

(ii) We have $B \preceq A$.

Proof. (i) $\Rightarrow$ (ii): Suppose that the set B is countable. Then we have $B \preceq \mathbb{N}_0$. Since the set A is countably infinite, we have $A \sim \mathbb{N}_0$. It follows from Proposition 4.6 that $B \preceq A$.

(ii) $\Rightarrow$ (i): Suppose that $B \preceq A$. It follows from $B \preceq A$ and $A \sim \mathbb{N}_0$ that $B \preceq \mathbb{N}_0$ (Proposition 4.6), that is, the set B is countable. □

Historical Notes:

Theorem 10.5 has been used by Richard Dedekind in [Dedekind 1888] to define what an infinite set is. For more details see Unit *The Natural Numbers and the Principle of Induction* [Garden 2020g].

11 Notes and References

Do you want to learn more? In Unit *Cardinal Arithmetic* [Garden 2020k] we will extend the elementary operations $m + n$, $m \cdot n$ and m^n from the natural numbers to general cardinal numbers. We will explain equations and inequalities like

$$a + a = a, \ a \cdot a = a \text{ and } 2^a > a \text{ for all infinite cardinal numbers } a.$$

12 Literature

A list of text books about set theory can be found at Literature about Set Theory.

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13 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

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