M. Garden

Cardinal Arithmetic

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Contents

1 Introduction

The present unit is part of the walk *The Cardinality of Sets* consisting of the following units:

1. Finite Sets and their Cardinalities [Garden 2020f]
2. Well-Ordered Sets [Garden 2020g]
3. Ordinal Numbers [Garden 2020h]
4. Cardinal Numbers [Garden 2020i]
5. Cardinal Arithmetic (this unit)
6. The Axiomatics of von Neumann, Bernays and Gödel [Garden 2020j]

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In Unit *Cardinal Numbers* [Garden 2020i] we have explained the cardinality $|A|$ of a set A. Its main properties are the facts that to every set A its cardinality $|A|$ may be defined and that we have

$$|A| = |B| \text{ if and only if } A \text{ and } B \text{ are equivalent,}$$

that is, if and only if there exists a bijective mapping $\alpha : A \to B$ from the set A onto the set B.

In the present unit we will deal with the question how to compute the cardinality of a set. For that purpose, we will look at the standard operations on sets, namely:

- the union of sets,
- the intersection of sets,
- the complement of a set,
- the direct product of sets,
- the power set $\mathcal{P}(A)$ of a set A,
- and the set $\mathcal{F}(A, B)$ of the functions $\alpha : A \to B$ from a set A into a set B.

In other words, we are going to investigate the cardinality of the sets $A \cup B$, $A \cap B$, $A \setminus B$, $A \times B$, $\mathcal{P}(A)$ and $\mathcal{F}(A, B)$ under the assumption that the cardinalities $|A|$ and $|B|$ of the sets A and B are known. It will turn out that we may extend the algebraic operations

$$m + n, \ m \cdot n \text{ and } m^n$$

defined on the natural numbers to arbitrary cardinal numbers and that these operations are closely related to the operations on sets described above. More precisely, for two sets A and B with $|A| = a$ and $|B| = b$ we will have

$$\begin{aligned} a + b &= |A \,\dot\cup\, B| \text{ if } A \cap B = \emptyset \\ a \cdot b &= |A \times B| \text{ and} \\ a^b &= |\mathcal{F}(B, A)|. \end{aligned}$$

If a and b are two infinite cardinal numbers, two main results of this unit are as follows:

$$a + b = a \times b = \max\{a, b\} \text{ and } 2^a > a.$$

The material of the present unit is organized as follows:

Elementary Properties of Cardinal Numbers (see Section 2):

In this section we will summarize the most important properties of cardinal numbers which have been explained in Unit *Cardinal Numbers* [Garden 2020i].

Extensions of Functions (see Section 3):

We will use several results about the extension of functions explained in former units. These results are summarized in this section.

Addition of Cardinal Numbers (see Section 4):

The real story of this unit starts in this section: We will define the addition of two cardinal numbers $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$ by

$$\mathfrak{a} + \mathfrak{b} := |A \mathbin{\dot\cup} B| \text{ if } A \cap B = \emptyset.$$

See Definition 4.2. Many properties which are well known from the addition of natural numbers remain valid: We have

$$(\mathfrak{a} + \mathfrak{b}) + \mathfrak{c} = \mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) \text{ and } \mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$$

for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ (see Theorem 4.4).

If $\mathfrak{a} \leqslant \mathfrak{c}$ and $\mathfrak{b} \leqslant \mathfrak{d}$, then we have $\mathfrak{a} + \mathfrak{b} \leqslant \mathfrak{c} + \mathfrak{d}$ for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ (see Theorem 4.6).

We have

$$|A| + |B| = |A \cup B| + |A \cap B|$$

for all sets A and B (see Theorem 4.12).

Multiplication of Cardinal Numbers (see Section 5):

We will define the product of two cardinal numbers $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$ by

$$\mathfrak{a} \cdot \mathfrak{b} := |A \times B|.$$

See Definition 5.1. Again, many properties which are well known from the multiplication of natural numbers remain valid: We have

$$(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}) \text{ and } \mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$$

for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ (see Theorem 5.7).

We have

$$\mathfrak{a} \cdot (\mathfrak{b} + \mathfrak{c}) = \mathfrak{a} \cdot \mathfrak{b} + \mathfrak{a} \cdot \mathfrak{c} \text{ and } (\mathfrak{a} + \mathfrak{b}) \cdot \mathfrak{c} = \mathfrak{a} \cdot \mathfrak{c} + \mathfrak{b} \cdot \mathfrak{c}$$

for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ (see Theorem 5.9).

If $\mathfrak{a} \leqslant \mathfrak{c}$ and $\mathfrak{b} \leqslant \mathfrak{d}$, then we have $\mathfrak{a} \cdot \mathfrak{b} \leqslant \mathfrak{c} \cdot \mathfrak{d}$ for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ (see Theorem 5.10).

Power of Cardinal Numbers (see Section 6):

We will define the power $\mathfrak{a}^{\mathfrak{b}}$ of two cardinal numbers $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$ by

$$\mathfrak{a}^{\mathfrak{b}} := |\mathcal{F}(B, A)|.$$

See Definition 6.1. Once again, many properties which are well known from the exponentiation of natural numbers remain valid: We have

$$\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}},\ (\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}} \text{ and } \left(\mathfrak{a}^{\mathfrak{b}}\right)^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$$

for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ (see Theorem 6.8).

If $\mathfrak{a} \leqslant \mathfrak{b}$, then we have $\mathfrak{a}^{\mathfrak{c}} \leqslant \mathfrak{b}^{\mathfrak{c}}$ for all cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ (see Theorem 6.9).

We have

$$|\mathcal{P}(A)| = 2^{\mathfrak{a}} \text{ and } \mathfrak{a} < 2^{\mathfrak{a}}$$

for all cardinal numbers $\mathfrak{a} = |A|$ (see Theorem 6.11).

Countable Sets (see Section 7):

A set A is called **countable** if the set A is finite or if we have $|A| = \aleph_0$. Countable sets have already been introduced in Unit *Finite Sets and their Cardinalities* [Garden 2020f]. Many results in this section prepare more general results explained in Section 8.

The main results are as follows: Let A be a countably infinite set, and let $\mathfrak{a} := |A|$.

We have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ (see Theorem 7.8).

We have $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ (see Theorem 7.6).

We have $|A| = \big|\{X \subseteq A \mid X \text{ finite}\}\big|$ (see Theorem 7.12).

Arithmetic of Infinite Cardinal Numbers (see Section 8):

We are no able to prove the main results of this unit which are important tools to compute the cardinality of a set: Let A be an infinite set, and let $\mathfrak{a} := |A|$.

We have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ (see Theorem 8.3).

We have $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ (see Theorem 8.7).

We have $|A| = \big|\{X \subseteq A \mid X \text{ finite}\}\big|$ (see Theorem 8.11).

Finally, we will show the Theorem of König and Zermelo saying that if $\mathfrak{a}_i < \mathfrak{b}_i$ for all elements i of an index set I, then we have

$$\sum_{i \in I} \mathfrak{a}_i < \prod_{i \in I} \mathfrak{b}_i$$

(see Theorem 8.12).

Summary of the Unit Cardinal Arithmetic (see Section 9):

For better clarity we summarize all results of this unit in this final section.

2 Elementary Properties of Cardinal Numbers

In Unit *Cardinal Numbers* [Garden 2020i] we have introduced the cardinal numbers. Note that a cardinal number is itself a set. In what follows we summarize the most important facts about cardinal numbers needed in this unit:

2.1 Definition. Two sets A and B are called **equivalent** if there exists a bijective function $\alpha : A \longrightarrow B$ from the set A onto the set B. If A and B are two equivalent sets, we write $A \sim B$.

For more details about functions and equivalent sets see Unit *Functions and Equivalent Sets* [Garden 2020b].

2.2 Theorem. *Let A be a set. Then there exists exactly one cardinal number $\mathfrak{a}$ such that $A \sim \mathfrak{a}$. In this case we write $|A| := \mathfrak{a}$.*

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.3 Theorem. *Let A and B be two sets. Then the following conditions are equivalent:*

(i) The sets A and B are equivalent.

(ii) We have $|A| = |B|$.

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.4 Theorem. *Let A and B be two sets. The following conditions are equivalent:*

(i) We have $|A| \leqslant |B|$.

(ii) There exists an injective mapping $\alpha : A \longrightarrow B$ from the set A into the set B.

(iii) There exists a surjective mapping $\beta : B \longrightarrow A$ from the set B onto the set A.

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.5 Proposition. *(a) Let A and B be two sets. If the set A is a subset of the set B, then we have $|A| \leqslant |B|$.*

(b) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers with $\mathfrak{a} \leqslant \mathfrak{b}$. Then there exist two sets A and B such that

$$|A| = \mathfrak{a},\ |B| = \mathfrak{b} \text{ and } A \subseteq B.$$

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.6 Theorem. (Cantor) *Let A be an arbitrary set. Then we have*

$$|A| < |\mathcal{P}(A)|$$

where $\mathcal{P}(A)$ denotes the power set of the set A.

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.7 Theorem. *Every natural number is a cardinal number.*

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

2.8 Theorem. *(a) The set $\mathbb{N}_0$ of the natural numbers is a cardinal number.*

(b) Let A be an infinite set. Then we have $\mathbb{N}_0 \leqslant |A|$.

Proof. See Unit *Cardinal Numbers* [Garden 2020i]. □

3 Extensions of Functions

In this unit we will need several results explained in former units about the extension of functions. These results look quite similar, but there are some differences. For the sake of clarity we summarize these results in this section:

3.1 Proposition. *Let I be an index set, let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of non-empty sets, and suppose that the sets $(A_i)_{i\in I}$ and the sets $(B_i)_{i\in I}$ are pairwise disjoint. For each element i of the set I let $\alpha_i : A_i \to B_i$ be a function from the set A_i into the set B_i.*

(a) There exists a function

$$\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$$

such that

$$\alpha(x) = \alpha_i(x) \text{ for all } x \in A_i \text{ and for all } i \in I.$$

(b) If the functions $\alpha_i : A_i \to B_i$ are injective for all elements i of the set I, then the function $\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also injective.

(c) If the functions $\alpha_i : A_i \to B_i$ are surjective for all elements i of the set I, then the function $\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also surjective.

(d) If the functions $\alpha_i : A_i \to B_i$ are bijective for all elements i of the set I, then the function $\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also bijective.

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020c]. □

3.2 Proposition. *Let I be an index set, and let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of non-empty sets. For each element i of the set I let $\alpha_i : A_i \to B_i$ be a function from the set A_i into the set B_i.*

Suppose that for each two elements i and j of the set I, we have $\alpha_i(x) = \alpha_j(x)$ for all elements x of the set $A_i \cap A_j$.[a]

(a) There exists a function

$$\alpha : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$$

such that $\alpha|_{A_i} = \alpha_i$ for all elements i of the set I.

(b) If the functions $\alpha_i : A_i \to B_i$ are surjective for all elements i of the set I, then the

function $\alpha : \bigcup_{i \in I} A_i \to \bigcup_{i \in I} B_i$ *is also surjective.*

[a] Note that this condition is automatically fulfilled if the sets A_i are pairwise disjoint.

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020c]. □

3.3 Proposition. *Let* I *be a totally ordered index set, and let* $(A_i)_{i \in I}$ *and* $(B_i)_{i \in I}$ *be two families of sets with the following properties:*

(i) The families $(A_i)_{i \in I}$ *and* $(B_i)_{i \in I}$ *are chains, that is, we have*

$$A_i \subseteq A_j \text{ and } B_i \subseteq B_j \text{ if } i \leqslant j.$$

(ii) For each element i *of the set* I*, there exists a function* $\alpha_i : A_i \to B_i$ *from the set* A_i *into the set* B_i*.*

(iii) For each two elements i *and* j *of the set* I *such that* $i \leqslant j$ *the function* $\alpha_i : A_i \to B_i$ *is induced by the function* $\alpha_j : A_j \to B_j$*, that is, we have*

$$\alpha_i(x) = \alpha_j(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i \in I} A_i \text{ and } B := \bigcup_{i \in I} B_i.$$

(a) There exists exactly one function $\alpha : A \to B$ *from the set* $A = \bigcup_{i \in I} A_i$ *into the set* $B = \bigcup_{i \in I} B_i$ *such that* $\alpha|_{A_i} = \alpha_i$ *for all elements* i *of the set* I*.*

(b) If the functions $\alpha_i : A_i \to B_i$ *are bijective for all elements* i *of the set* I*, then the function* $\alpha : A \to B$ *is also bijective.*

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

3.4 Proposition. *Let* I *be an index set, and let* $(A_i)_{i \in I}$ *and* $(B_i)_{i \in I}$ *be two families of sets. Suppose that for each element* i *of the set* I *there exists a bijective function* $\alpha_i : A_i \to B_i$ *from the set* A_i *onto the set* B_i*.*

Then there exists a bijective function

$$\alpha : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

from the direct product $\prod_{i \in I} A_i$ *onto the direct product* $\prod_{i \in I} B_i$*.*

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020c]. □

4 Addition of Cardinal Numbers

We have seen in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] that we have

$$m + n = |A \cup B|$$

for each two finite disjoint sets A and B with $|A| = m$ and $|B| = n$. We will use this property in Definition 4.2 to define the sum of two arbitrary cardinal numbers.

The main results of this section are that the so-defined addition is associative and commutative (Theorem 4.4) , that we have $a + c \leqslant b + d$ if $a \leqslant c$ and $b \leqslant d$ (Theorem 4.6) and that

$$|A| + |B| = |A \cup B| + |A \cap B|$$

for each two sets A and B (Theorem 4.12).

Definition of the Addition of Cardinal Numbers:

In Definition 4.2 we will define the sum of two cardinal numbers. In order to verify that this definition is well-defined we need Proposition 4.1.

4.1 Proposition. *Let A, B, C and D be four sets such that $A \cap B = \emptyset$ and $C \cap D = \emptyset$. If $A \sim C$ and $B \sim D$, then we have $A \cup B \sim C \cup D$.*

Proof. Since $A \sim C$ and $B \sim D$, there exist two bijective functions $\alpha : A \to C$ and $\beta : B \to D$ from the set A onto the set C and from the set B onto the set D, respectively. By Proposition 3.1, there exists a bijective function $\gamma : A \cup B \to C \cup D$ from the set $A \cup B$ onto the set $C \cup D$. It follows that $A \cup B \sim C \cup D$. □

4.2 Definition. Let a and b be two cardinal numbers, and let A and B be two sets such that $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$. Then the cardinal number $|A \cup B|$ is called **the sum of the cardinal numbers a and b**. It is denoted by $a + b$.

French / German. Sum of cardinal numbers = Somme de nombres cardinaux = Summe von Kardinalzahlen.

4.3 Remarks. (a) In view of Proposition 4.1 the definition of the sum $a + b = |A \cup B|$ is well-defined, that is, independent from the concrete choice of the sets A and B such that $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$.

(b) For two cardinal numbers a and b, the sets $A := a \times \{0\}$ and $B := b \times \{1\}$ fulfill the conditions $|A| = a$, $|B| = b$ and $A \cap B = \emptyset$.

(c) Since each natural number is a cardinal number (Theorem 2.7), we also have defined the sum $n + m$ for all natural numbers n and m. In Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] we have given an alternative definition of the sum of two natural numbers. In Unit *Finite Sets and their Cardinalities* [Garden 2020f] we have seen that this definition is equivalent to Definition 4.2.

General Properties of the Addition of Cardinal Numbers:

4.4 Theorem. *Let a, b and c be three cardinal numbers.*

(a) We have $(a + b) + c = a + (b + c)$ (addition is associative).

(b) We have $a + b = b + a$ (addition is commutative).

Proof. Let A, B and C be three pairwise disjoint sets such that $|A| = a$, $|B| = b$ and $|C| = c$.

(a) We have $(a + b) + c = |(A \cup B) \cup C| = |A \cup (B \cup C)| = a + (b + c)$.

(b) We have $a + b = |A \cup B| = |B \cup A| = b + a$. □

4.5 Proposition. *Let* $\mathfrak{a}$ *be a cardinal number. Then we have* $\mathfrak{a}+0=\mathfrak{a}=0+\mathfrak{a}$.

Proof. Since $|\mathfrak{a}|=\mathfrak{a}$ and $|\emptyset|=0$, we have $\mathfrak{a}+0=|\mathfrak{a}\,\dot\cup\,\emptyset|=|\mathfrak{a}|=\mathfrak{a}$. By Theorem 4.4, we have $0=\mathfrak{a}+0=0+\mathfrak{a}$. □

4.6 Theorem. *Let* $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *and* $\mathfrak{d}$ *be four cardinal numbers such that* $\mathfrak{a}\leqslant\mathfrak{b}$ *and* $\mathfrak{c}\leqslant\mathfrak{d}$. *Then we have*

$$\mathfrak{a}+\mathfrak{c}\leqslant\mathfrak{b}+\mathfrak{d}.$$

Proof. Since $\mathfrak{a}\leqslant\mathfrak{b}$ and $\mathfrak{c}\leqslant\mathfrak{d}$, it follows from Proposition 2.5 that there exist four sets A, B, C and D such that $|A|=\mathfrak{a}$, $|B|=\mathfrak{b}$, $|C|=\mathfrak{c}$, $|D|=\mathfrak{d}$, $A\subseteq B$ and $C\subseteq D$. W.lo.g. we may suppose that $B\cap D=\emptyset$ (if not replace the sets A and B by $A\times\{0\}$ and $B\times\{0\}$ and the sets C and D by $C\times\{1\}$ and $D\times\{1\}$).

It follows that $A\cup C\subseteq B\cup D$. By Proposition 2.5, we get

$$\mathfrak{a}+\mathfrak{c}=|A\,\dot\cup\,C|\leqslant|B\,\dot\cup\,D|=\mathfrak{b}+\mathfrak{d}.$$

□

4.7 Definition. Let I be an index set, let $(\mathfrak{a}_i)_{i\in I}$ be a family of cardinal numbers, and let $(A_i)_{i\in I}$ be a family of pairwise disjoint sets such that $|A_i|=\mathfrak{a}_i$ for all elements i of the set I.

Then the cardinal number $|\bigcup_{i\in I}A_i|$ is called the **sum of the cardinal numbers** $\mathfrak{a}_i$. It is denoted by

$$\sum_{i\in I}\mathfrak{a}_i:=|\bigcup_{i\in I}A_i|.$$

4.8 Proposition. *Let* I *be an index set, and let* $(\mathfrak{a}_i)_{i\in I}$ *be a family of cardinal numbers. Then the sum* $\sum_{i\in I}\mathfrak{a}_i$ *is well-defined.*

Proof. Let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of pairwise disjoint sets such that $|A_i|=\mathfrak{a}_i$ and $|B_i|=\mathfrak{a}_i$ for all elements i of the set I. Let i be an element of the set I. Since $|A_i|=\mathfrak{a}_i=|B_i|$, it follows that there exists a bijective function $\alpha_i:A_i\longrightarrow B_i$ from the set A_i onto the set B_i. By Proposition 3.1, there exists a bijective function

$$\alpha:\bigcup_{i\in I}A_i\to\bigcup_{i\in I}B_i$$

implying that $|\bigcup_{i\in I}A_i|=|\bigcup_{i\in I}B_i|$. □

4.9 Proposition. *Let* I *be an index set, and let* $(\mathfrak{a}_i)_{i\in I}$ *and* $(\mathfrak{b}_i)_{i\in I}$ *be two families of cardinal numbers such that*

$$\mathfrak{a}_i\leqslant\mathfrak{b}_i \textit{ for all } i\in I.$$

Then we have

$$\sum_{i\in I}\mathfrak{a}_i\leqslant\sum_{i\in I}\mathfrak{b}_i.$$

Proof. By Proposition 2.5, for each element i of the set I there exist two sets A_i and B_i such that

$$|A_i| = a_i,\ |B_i| = b_i \text{ and } A_i \subseteq B_i.$$

For each element i of the set I let $A_i' := A_i \times \{i\}$ and $B_i' := B_i \times \{i\}$. Then we have

$$A_i' \cap A_j' = \emptyset \text{ and } B_i' \cap B_j' = \emptyset \text{ for all } i, j \in I \text{ with } i \neq j.$$

In addition, the functions $\alpha_i : A_i' \to B_i'$ defined by $\alpha_i : (x, i) \mapsto (x, i)$ are injective.

By Proposition 3.1, there exists an injective function

$$\alpha : \bigcup_{i \in I} A_i' \to \bigcup_{i \in I} B_i'.$$

It follows from Theorem 2.4 that

$$\sum_{i \in I} a_i = \Big|\bigcup_{i \in I} A_i'\Big| \leqslant \Big|\bigcup_{i \in I} B_i'\Big| = \sum_{i \in I} b_i.$$

□

4.10 Proposition. *Let I be an index set, and let $(A_i)_{i \in I}$ be a family of sets. Then we have*

$$\Big|\bigcup_{i \in I} A_i\Big| \leqslant \sum_{i \in I} |A_i|.$$

In particular, we have $|A \cup B| \leqslant |A| + |B|$ for all sets A and B.

Proof. For each element i of the set I, let $\alpha_i : A_i \times \{i\} \to A_i$ be the function defined by $\alpha_i : (x, i) \mapsto x$. Obviously, the function $\alpha_i : A_i \times \{i\} \to A_i$ is bijective and in particular surjective for all elements i of the set I.

Since the sets $A_i \times \{i\}$ are pairwise disjoint, it follows from Proposition 3.2 that there exists a surjective function

$$\alpha : \bigcup_{i \in I} (A_i \times \{i\}) \to \bigcup_{i \in I} A_i.$$

It follows from Theorem 2.4 that

$$\Big|\bigcup_{i \in I} A_i\Big| \leqslant \Big|\bigcup_{i \in I} (A_i \times \{i\})\Big| = \sum_{i \in I} |A_i|.$$

□

4.11 Proposition. *Let A and B be two sets. Then we have*

$$|A| = |A \setminus (A \cap B)| + |A \cap B| = |A \setminus B| + |A \cap B|.$$

Proof. Let A and B be two sets. Obviously, we have

$$A \setminus (A \cap B) = A \setminus B \text{ and } A = \big(A \setminus (A \cap B)\big) \mathbin{\dot\cup} (A \cap B).$$

By Definition 4.2, we get

$$|A| = |A \setminus (A \cap B)| + |A \cap B| = |A \setminus B| + |A \cap B|.$$

□

4.12 Theorem. *Let* A *and* B *be two sets. Then we have*

$$|A| + |B| = |A \cup B| + |A \cap B|.$$

Proof. Let A and B be two sets. Note that

$$A \cup B = \big(A \setminus (A \cap B)\big) \mathbin{\dot\cup} B.$$

It follows from Proposition 4.11 that

$$\begin{aligned} |A| + |B| &= \big(|A \setminus (A \cap B)| + |A \cap B|\big) + |B| \\ &= |A \cap B| + \big(|A \setminus (A \cap B)| + |B|\big) = |A \cap B| + |A \cup B|. \end{aligned}$$

□

4.13 Remark. In Corollary 8.5 we will see that we even have $|A \cup B| = |A| + |B|$ if at least one of the two sets A or B is infinite.

4.14 Proposition. *Let* A *be an infinite set, and let* B *be a finite subset of the set* A*. Then the set* $A \setminus B$ *is infinite.*

Proof. Let $n := |B|$. Assume that the set $A \setminus B$ is finite, and let $m := |A \setminus B|$. By Proposition 4.11, we have

$$|A| = |A \setminus B| + |A \cap B| \leqslant |A \setminus B| + |B| = m + n.$$

Since the sum $m + n$ is a natural number, it follows that the set A is finite, a contradiction. □

Historical Notes:

The idea to define the sum $a + b$ of two cardinal numbers $a = |A|$ and $b = |B|$ by the formula

$$|A| + |B| = |A \mathbin{\dot\cup} B| \text{ if } A \cap B = \emptyset$$

(see Definition 4.2) is due to Georg Cantor:

Die Vereinigung zweier Mengen M *und* N*, die keine gemeinsamen Elemente haben, wurde [...] mit* (M, N) *bezeichnet. [...] Dies führt zur Definition der Summe von* a *und* b*, indem wir setzen:*

$$a + b = \overline{\overline{(M, N)}}.$$

See [Cantor 1895, p. 485].

The union of two sets M *and* N *which do not have any elements in common has been denoted [...] by* (M, N)*. [...] This yields the definition of the sum of* a *and* b *by defining*

$$a + b = \overline{\overline{(M, N)}}.$$

Cantor uses the notation $\overline{\overline{A}}$ for $|A|$ and (M, N) for $M \cup N$. So the equation $a + b = \overline{\overline{(M, N)}}$ means $a + b = |A \cup B|$. The equations $a + b = b + a$ and $a + (b + c) = (a + b) + c$ are also due to Cantor:

Da im Mächtigkeitsbegriff von der Ordnung der Elemente abstrahiert ist, so folgt ohne Weiteres

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$$

und für je drei Kardinalzahlen $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}.$$

See [Cantor 1895, p. 485].

Since the concept of cardinality abstracts from the order of the elements, it follows immediately that

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{b} + \mathfrak{a}$$

and that for each three cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *we have*

$$\mathfrak{a} + (\mathfrak{b} + \mathfrak{c}) = (\mathfrak{a} + \mathfrak{b}) + \mathfrak{c}.$$

5 Multiplication of Cardinal Numbers

We have seen in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] that we have

$$m \cdot n = |A \times B|$$

for each two finite sets A and B with $|A| = m$ and $|B| = n$. We will use this property in Definition 5.1 to define the product of two arbitrary cardinal numbers.

The main results of this section are that the so-defined multiplication is associative and commutative (Theorem 5.7), that the distributive laws hold (Theorem 5.9) and that we have $\mathfrak{a} \cdot \mathfrak{c} \leqslant \mathfrak{b} \cdot \mathfrak{d}$ if $\mathfrak{a} \leqslant \mathfrak{c}$ and $\mathfrak{b} \leqslant \mathfrak{d}$ (Theorem 5.10).

Definition of the Product of Cardinal Numbers:

5.1 Definition. (a) Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers, and let A and B be two sets such that $|A| = \mathfrak{a}$ and $|B| = \mathfrak{b}$. Then the cardinal number $|A \times B|$ is called **the product of the cardinal numbers $\mathfrak{a}$ and $\mathfrak{b}$**. It is denoted by

$$\mathfrak{a} \cdot \mathfrak{b} := |A \times B|.$$

Alternatively, we also write $\mathfrak{a}\mathfrak{b}$ instead of $\mathfrak{a} \cdot \mathfrak{b}$.

(b) Let I be an index set, let $(\mathfrak{a}_i)_{i \in I}$ be a family of cardinal numbers, and let $(A_i)_{i \in I}$ be a family of sets such that $|A_i| = \mathfrak{a}_i$ for all elements i of the set I.

Then the cardinal number $|\prod_{i \in I} A_i|$ is called the **product of the cardinal numbers** $\mathfrak{a}_i$. It is denoted by

$$\prod_{i \in I} \mathfrak{a}_i := |\prod_{i \in I} A_i|.$$

French / German. Product of cardinal numbers = Produit des nombres cardinaux = Produkt von Kardinalzahlen.

5.2 Remark. Since each natural number is a cardinal number (Theorem 2.7), we also have defined the product $m \cdot n$ for all natural numbers m and n. In Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] we have given an alternative definition of the product of two natural numbers. In Unit *Finite Sets and their Cardinalities* [Garden 2020f] we have seen that this definition is equivalent to Definition 5.1.

5.3 Proposition. *Let I be an index set, let $(a_i)_{i \in I}$ be a family of cardinal numbers. Then the product $\prod_{i \in I} a_i$ is well-defined.*

Proof. Let $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be two families of sets such that $|A_i| = a_i$ and $|B_i| = a_i$ for all elements i of the set I. Let i be an element of the set I. Since $|A_i| = a_i = |B_i|$, it follows that there exists a bijective function $\alpha_i : A_i \to B_i$ from the set A_i onto the set B_i. By Proposition 3.4, there exists a bijective function

$$\alpha : \prod_{i \in I} A_i \to \prod_{i \in I} B_i$$

implying that $|\prod_{i \in I} A_i| = |\prod_{i \in I} B_i|$. □

General Properties of the Product of Cardinal Numbers:

For the proof of Proposition 5.5 we need the following result about direct products:

5.4 Proposition. *Let A and B be two sets. Then we have*

$$A \times B = \emptyset \text{ if and only if } A = \emptyset \text{ or } B = \emptyset.$$

Proof. See Unit *Direct Products and Relations* [Garden 2020a]. □

5.5 Proposition. *Let a and b be two cardinal numbers.*

(a) We have $a \cdot 0 = 0 \cdot a = 0$.

(b) We have $a \cdot b = 0$ if and only if $a = 0$ or $b = 0$.

(c) We have $a \cdot 1 = 1 \cdot a = a$.

Proof. (a) By Proposition 5.4, we have $A \times \emptyset = \emptyset \times A = \emptyset$. It follows that $a \cdot 0 = 0 \cdot a = 0$.
(b) Note that we have $|X| = 0$ if and only if $X = \emptyset$ for all sets X. Let A and B be two sets such that $|A| = a$ and $|B| = b$. Then we have

$$\begin{aligned} & 0 = a \cdot b = |A \times B| \Leftrightarrow \emptyset = A \times B \\ \Leftrightarrow \quad & A = \emptyset \text{ or } B = \emptyset \text{ (Proposition 5.4)} \\ \Leftrightarrow \quad & a = 0 \text{ or } b = 0. \end{aligned}$$

(c) The mappings $\alpha : A \to A \times \{1\}$ and $\beta : A \to \{1\} \times A$ defined by $\alpha : x \mapsto (x, 1)$ and $\beta : x \mapsto (1, x)$ are bijective. It follows that

$$a = |A| = |A \times \{1\}| = a \cdot 1 \text{ and } a = |A| = |\{1\} \times A| = 1 \cdot a.$$

□

5.6 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers, let I be an index set such that $|I| = \mathfrak{b}$, and let $\mathfrak{a}_i := \mathfrak{a}$ for all elements i of the set I. Then we have*

$$\mathfrak{a} \cdot \mathfrak{b} = \sum_{i \in I} \mathfrak{a}_i.$$

In particular, we have $\mathfrak{a} + \mathfrak{a} = 2\mathfrak{a}$ and

$$\underbrace{\mathfrak{a} + \ldots + \mathfrak{a}}_{n} = n\mathfrak{a}.$$

Proof. Let A be a set such that $|A| = \mathfrak{a}$. Then we have

$$\mathfrak{a} \cdot \mathfrak{b} = |A \times I| = \Big|\bigcup_{i \in I} A \times \{i\}\Big| = \sum_{i \in I} \mathfrak{a}_i.$$

□

5.7 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be three cardinal numbers.*

(a) We have $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c})$ (multiplication is associative).

(b) We have $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$ (multiplication is commutative).

Proof. (a) Let A_1, A_2 and A_3 be three sets such that $|A_1| = \mathfrak{a}$, $|A_2| = \mathfrak{b}$ and $|A_3| = \mathfrak{c}$, let $I := \{1, 2, 3\}$, and let $A := \prod_{i \in I} A_i$. Let

$$\alpha_1 : A \longrightarrow (A_1 \times A_2) \times A_3 \text{ be defined by } \alpha_1 : (a_1, a_2, a_3) \longmapsto \big((a_1, a_2), a_3\big),$$

and let

$$\alpha_2 : A \longrightarrow A_1 \times (A_2 \times A_3) \text{ be defined by } \alpha_2 : (a_1, a_2, a_3) \longmapsto \big((a_1, (a_2, a_3)\big).$$

The functions $\alpha_1 : A \longrightarrow (A_1 \times A_2) \times A_3$ and $\alpha_2 : A \longrightarrow A_1 \times (A_2 \times A_3)$ are obviously bijective implying that

$$(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} = |(A_1 \times A_2) \times A_3| = |A| = |A_1 \times (A_2 \times A_3)| = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}).$$

(b) Let A and B be two sets such that $|A| = \mathfrak{a}$ and $|B| = \mathfrak{b}$. Obviously, the mapping $\alpha : A \times B \longrightarrow B \times A$ defined by $\alpha : (a, b) \longmapsto (b, a)$ is bijective implying that

$$\mathfrak{a} \cdot \mathfrak{b} = |A \times B| = |B \times A| = \mathfrak{b} \cdot \mathfrak{a}.$$

□

We will need the following elementary result about direct products in the proof of Theorem 5.9:

5.8 Proposition. *Let A, B and C be three sets.*

(a) We have $A \times (B \cup C) = (A \times B) \cup (A \times C)$.

(b) We have $A \times (B \cap C) = (A \times B) \cap (A \times C)$.

Proof. See Unit *Direct Products and Relations* [Garden 2020a]. □

5.9 Theorem. (Distributive Laws) *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be three cardinal numbers.*

(a) We have $\mathfrak{a}(\mathfrak{b}+\mathfrak{c}) = \mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}$.

(b) We have $(\mathfrak{a}+\mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c}+\mathfrak{b}\mathfrak{c}$.

Proof. (a) Let A, B and C be three sets such that $|A| = \mathfrak{a}$, $|B| = \mathfrak{b}$, $|C| = \mathfrak{c}$ and $B \cap C = \emptyset$. It follows from Proposition 5.8 and Proposition 5.4 that

$$(A \times B) \cap (A \times C) = A \times (B \cap C) = A \times \emptyset = \emptyset.$$

Again by Proposition 5.8 and Proposition 5.4, we get

$$\mathfrak{a}(\mathfrak{b}+\mathfrak{c}) = |A \times (B \uplus C)| = |(A \times B) \uplus (A \times C)| = |A \times B| + |A \times C| = \mathfrak{a}\mathfrak{b}+\mathfrak{a}\mathfrak{c}.$$

(b) follows from (a) and the commutative law for multiplication (Theorem 5.7). □

5.10 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ be four cardinal numbers such that $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{c} \leqslant \mathfrak{d}$. Then we have*

$$\mathfrak{a} \cdot \mathfrak{c} \leqslant \mathfrak{b} \cdot \mathfrak{d}.$$

Proof. Since $\mathfrak{a} \leqslant \mathfrak{b}$ and $\mathfrak{c} \leqslant \mathfrak{d}$, it follows from Proposition 2.5 that there exist four sets A, B, C and D such that $|A| = \mathfrak{a}$, $|B| = \mathfrak{b}$, $|C| = \mathfrak{c}$, $|D| = \mathfrak{d}$, $A \subseteq B$ and $C \subseteq D$. It follows that $A \times C \subseteq B \times D$. By Proposition 2.5, we get

$$\mathfrak{a} \cdot \mathfrak{c} = |A \times C| \leqslant |B \times D| = \mathfrak{b} \cdot \mathfrak{d}.$$

□

Historical Notes:

The idea to define the product $\mathfrak{a} \cdot \mathfrak{b}$ of two cardinal numbers $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$ by the formula

$$|A| \cdot |B| = |A \times B|$$

(see Definition 5.1) is due to Georg Cantor:

Jedes Element m einer Menge M lässt sich mit jedem Element n einer anderen Menge N zu einem neuen Element (m, n) verbinden; für die Menge aller dieser Verbindungen (m, n) setzen wir die Bezeichnung $(M.N)$ fest. Wir nennen sie die **Verbindungsmenge von M und N**. *Es ist also*

$$(M.N) = \{(m, n)\}.$$

[...] Wir definieren nun das Produkt $\mathfrak{a} \cdot \mathfrak{b}$ durch die Gleichung

$$\mathfrak{a} \cdot \mathfrak{b} = \overline{\overline{(M.N)}}.$$

See [Cantor 1895, p. 485].

Every element m of a set M can be combined with every element n of another set N to form a new element (m, n); we denote the set of all these combinations (m, n) by $(M.N)$. We call it **the set of combinations of M and N**. *Hence, we have*

$$(M.N) = \{(m, n)\}.$$

[...] We now define the product $\mathfrak{a} \cdot \mathfrak{b}$ *by the equation*

$$\mathfrak{a} \cdot \mathfrak{b} = \overline{\overline{(M.N)}}.$$

In modern terminology we have $(M.N) = M \times N$ and $\mathfrak{a} \cdot \mathfrak{b} = |M \times N|$ where $|M| = \mathfrak{a}$ and $|N| = \mathfrak{b}$. Cantor was also aware of the equation

$$\mathfrak{a} \cdot \mathfrak{b} = \sum_{i \in I} \mathfrak{a}_i$$

where $|I| = \mathfrak{b}$ and $\mathfrak{a}_i = \mathfrak{a}$ for all elements i of the set I (see Proposition 5.6):

Eine Menge mit der Kardinalzahl $\mathfrak{a} \cdot \mathfrak{b}$ *lässt sich aus zwei Mengen* M *und* N *mit den Kardinalzahlen* $\mathfrak{a}$ *und* $\mathfrak{b}$ *auch nach folgender Regel herstellen: Man gehe von der Menge* N *aus und ersetze in ihr jedes Element* n *durch eine Menge* $M_n \sim M$*; fasst man die Elemente aller dieser Mengen* M_n *zu einem Ganzen* S *zusammen, so sieht man leicht, dass*

$$S \sim (M.N),$$

folglich

$$\overline{\overline{S}} = \mathfrak{a} \cdot \mathfrak{b}.$$

See [Cantor 1895, p. 486].

A set with the cardinal number $\mathfrak{a} \cdot \mathfrak{b}$ *can also be constructed from two sets* M *and* N *with the cardinal numbers* $\mathfrak{a}$ *and* $\mathfrak{b}$ *in the following way: One takes a set* N *and replaces every element* n *of this set by a set* $M_n \sim M$*; if we combine the elements of all these sets* M_n *into a whole* S*, then one easily sees that*

$$S \sim (M.N),$$

hence

$$\overline{\overline{S}} = \mathfrak{a} \cdot \mathfrak{b}.$$

One should be a little bit more precise by saying that the sets M_n have to be pairwise disjoint. Then we get

$$\mathfrak{a} \cdot \mathfrak{b} = \Big| \bigcup_{n \in N} M_n \Big| = \sum_{n \in N} |M_n|.$$

As a next step Cantor proves the equations $(\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c} = \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c})$ and $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} \cdot \mathfrak{a}$ (Theorem 5.7) and the distributive laws $\mathfrak{a}(\mathfrak{b} + \mathfrak{c}) = \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}$ and $(\mathfrak{a} + \mathfrak{b})\mathfrak{c} = \mathfrak{a}\mathfrak{c} + \mathfrak{b}\mathfrak{c}$ (Theorem 5.9):

Aus unseren Definitionen folgen leicht die Sätze:

$$\begin{aligned} \mathfrak{a} \cdot \mathfrak{b} &= \mathfrak{b} \cdot \mathfrak{a}, \\ \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}) &= (\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c}, \\ \mathfrak{a}(\mathfrak{b} + \mathfrak{c}) &= \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}. \end{aligned}$$

See [Cantor 1895, p. 486].

The following sentences follow easily from our definitions:

$$\begin{aligned} \mathfrak{a} \cdot \mathfrak{b} &= \mathfrak{b} \cdot \mathfrak{a}, \\ \mathfrak{a} \cdot (\mathfrak{b} \cdot \mathfrak{c}) &= (\mathfrak{a} \cdot \mathfrak{b}) \cdot \mathfrak{c}, \\ \mathfrak{a}(\mathfrak{b} + \mathfrak{c}) &= \mathfrak{a}\mathfrak{b} + \mathfrak{a}\mathfrak{c}. \end{aligned}$$

6 Power of Cardinal Numbers

We have seen in Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] that we have

$$\mathfrak{m}^{\mathfrak{n}} = \big|\{\alpha : B \longrightarrow A \mid \alpha \text{ function}\}\big|$$

for each two finite sets A and B with $|A| = \mathfrak{m}$ and $|B| = \mathfrak{n}$. We will use this property in Definition 6.1 to define the power of two arbitrary cardinal numbers.

The main results of this section are that we have $\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}$, $(\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}}$ and $\left(\mathfrak{a}^{\mathfrak{b}}\right)^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$ for each three cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ (Theorem 6.8), that $\mathfrak{a}^{\mathfrak{c}} \leqslant \mathfrak{b}^{\mathfrak{c}}$ if $\mathfrak{a} \leqslant \mathfrak{b}$ (Theorem 6.9), that $|\mathcal{P}(A)| = 2^{\mathfrak{a}}$ for all sets A with $|A| = \mathfrak{a}$ and that $\mathfrak{a} < 2^{\mathfrak{a}}$ for all cardinal numbers $\mathfrak{a}$ (Theorem 6.11).

Definition of the Power of Cardinal Numbers:

6.1 Definition. Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers, and let A and B be two sets such that $|A| = \mathfrak{a}$ and $|B| = \mathfrak{b}$. Let $\mathcal{F}(B, A)$ be the set of the functions $\alpha : B \longrightarrow A$ from the set B into the set A. Then the cardinal number

$$|\mathcal{F}(B, A)|$$

is called **the $\mathfrak{b}^{\text{th}}$ power of $\mathfrak{a}$**. It is denoted by $\mathfrak{a}^{\mathfrak{b}}$.

French / German. Power of a cardinal number = Puissance d'un nombre cardinal = Potenz einer Kardinalzahl.

6.2 Remark. Since each natural number is a cardinal number (Theorem 2.7), we also have defined the power $\mathfrak{m}^{\mathfrak{n}}$ for all natural numbers $\mathfrak{m}$ and $\mathfrak{n}$. In Unit *The Natural Numbers and the Principle of Induction* [Garden 2020e] we have given an alternative definition of the power of two natural numbers. In Unit *Finite Sets and their Cardinalities* [Garden 2020f] we have seen that this definition is equivalent to Definition 6.1.

6.3 Proposition. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers. Then the cardinal number $\mathfrak{a}^{\mathfrak{b}}$ is well-defined.*

Proof. Let A, B, C and D be four sets such that $|A| = |C| = \mathfrak{a}$ and $|B| = |D| = \mathfrak{b}$.

We have to show that there exists a bijective function from the set

$$\mathcal{F}(B, A) := \{\alpha : B \longrightarrow A \mid \alpha \text{ is a function from } B \text{ to } A\}$$

onto the set

$$\mathcal{F}(D, C) := \{\beta : D \longrightarrow C \mid \beta \text{ is a function from } D \text{ to } C\} :$$

$$\begin{tikzcd} A \arrow[r, "\gamma"] & C \\ B \arrow[u, "\alpha"'] \arrow[r, "\delta"] & D \arrow[u, "\beta"'] \end{tikzcd}$$

Since we have $|A| = |C| = \mathfrak{a}$ and $|B| = |D| = \mathfrak{b}$, there exist bijective functions $\gamma : A \longrightarrow C$ and $\delta : B \longrightarrow D$ from the set A onto the set C and from the set B onto the set D, respectively.

Define the functions

$$
\begin{aligned}
f &: \mathcal{F}(B,A) \to \mathcal{F}(D,C) \quad \text{by} \quad f : \alpha \mapsto \gamma \circ \alpha \circ \delta^{-1} \text{ and} \\
g &: \mathcal{F}(D,C) \to \mathcal{F}(B,A) \quad \text{by} \quad g : \beta \mapsto \gamma^{-1} \circ \beta \circ \delta.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
f(g(\beta)) &= f(\gamma^{-1} \circ \beta \circ \delta) = \gamma \circ \gamma^{-1} \circ \beta \circ \delta \circ \delta^{-1} = \beta \\
g(f(\alpha)) &= g(\gamma \circ \alpha \circ \delta^{-1}) = \gamma^{-1} \circ \gamma \circ \alpha \circ \delta^{-1} \circ \delta = \alpha.
\end{aligned}
$$

Hence, we have $f = g^{-1}$ and $g = f^{-1}$ implying that the function

$$f : \mathcal{F}(B,A) \to \mathcal{F}(D,C)$$

is bijective. □

Elementary Properties of the Power of Cardinal Numbers:

We will need the following result about functions in the proof of Proposition 6.5:

6.4 Proposition. *Let $\alpha : A \to B$ be a function. If $A = \emptyset$, then we have $\alpha = \emptyset$.*

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020b]. □

6.5 Proposition. *Let a be a cardinal number. Then we have $a^0 = 1$.*

Proof. Let A be a set such that $|A| = a$. By Proposition 6.4, we get

$$a^0 = |\{\alpha : \emptyset \to A \mid \alpha \text{ function}\}| = |\{\emptyset\}| = 1.$$

□

6.6 Proposition. *Let a and b be two cardinal numbers, and let A and I be two sets such that $|A| = a$ and $|I| = b$. For each element i of the set I, let $a_i := a$. Then we have*

$$a^b = \prod_{i \in I} a_i.$$

In particular, we have $a \cdot a = a^2$ and $\underbrace{a \cdot \ldots \cdot a}_{n} = a^n$.

Proof. Let B be a set such that $|B| = b$. Since $|I| = b$, we may write the set B in the form $B = \{b_i \mid i \in I\}$.

For each element i of the set I, let $A_i := A$, and let $\mathcal{F}(B,A)$ be the set of the functions $\alpha : B \to A$ from the set B into the set A. Define the function $\gamma : \mathcal{F}(B,A) \to \prod_{i \in I} A_i$ by

$$\gamma : \alpha \mapsto \big(\alpha(b_i)\big)_{i \in I}.$$

Step 1. *The function $\gamma : \mathcal{F}(B,A) \to \prod_{i \in I} A_i$ is injective:*

For, let α and β be two functions from the set $\mathcal{F}(B,A)$ such that $\gamma(\alpha) = \gamma(\beta)$. It follows that $\big(\alpha(b_i)\big)_{i \in I} = \big(\beta(b_i)\big)_{i \in I}$ implying that $\alpha(b_i) = \beta(b_i)$ for all elements i of the set I. It follows that $\alpha = \beta$.

Step 2. *The function* $\gamma : \mathcal{F}(B, A) \to \prod_{i\in I} A_i$ *is surjective:*

For, let $(a_i)_{i\in I}$ be an element of the product $\prod_{i\in I} A_i$. For each element i of the set I, set $\alpha(b_i) := a_i$. Then we obtain a function $\alpha : B \to A$ such that $\gamma(\alpha) = (a_i)_{i\in I}$.

Step 3. We have $\mathfrak{a}^{\mathfrak{b}} = \prod_{i\in I} \mathfrak{a}_i$:

It follows from Step 1 and 2 that $\mathfrak{a}^{\mathfrak{b}} = |\mathcal{F}(B, A)| = |\prod_{i\in I} A_i| = \prod_{i\in I} \mathfrak{a}_i$. □

We will need the following result about direct products in the proof of Theorem 6.8:

6.7 Proposition. *Let I and J be two disjoint index sets, and let A_k be a set for each element k of the set $I \cup J$. Then we have*

$$\prod_{k\in I\dot\cup J} A_k \sim \Big(\prod_{i\in I} A_i\Big) \times \Big(\prod_{j\in J} A_j\Big).$$

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020c]. □

6.8 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be three cardinal numbers.*

(a) We have $\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}$.

(b) We have $(\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}}$.

(c) We have $\big(\mathfrak{a}^{\mathfrak{b}}\big)^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$.

Proof. (a) Let A, B and C be three sets such that $|A| = \mathfrak{a}$, $|B| = \mathfrak{b}$, $|C| = \mathfrak{c}$ and $B \cap C = \emptyset$. For all elements i of the set $B \dot\cup C$, let $A_i := A$. It follows from Proposition 6.6, Proposition 6.7 and Theorem 2.3 that

$$\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}} = |\prod_{i\in B\dot\cup C} A_i| = |\prod_{i\in B} A_i \times \prod_{j\in C} A_j| = |\prod_{i\in B} A_i| \cdot |\prod_{j\in C} A_j| = \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}}.$$

(b) For two sets X and Y, let $\mathcal{F}(X, Y)$ be the set of the functions from the set X into the set Y. We have to show that

$$\mathcal{F}(C, A \times B) \sim \mathcal{F}(C, A) \times \mathcal{F}(C, B).$$

For, define the function $\gamma : \mathcal{F}(C, A) \times \mathcal{F}(C, B) \to \mathcal{F}(C, A \times B)$ as follows:

$$\gamma : (\alpha, \beta) \mapsto \Big(\delta : C \to A \times B \text{ defined by } \delta : x \mapsto (\alpha(x), \beta(x))\Big).$$

Step 1. *The function* $\gamma : \mathcal{F}(C, A) \times \mathcal{F}(C, B) \to \mathcal{F}(C, A \times B)$ *is injective:*

For, let α_1 and α_2 be two functions of the set $\mathcal{F}(C, A)$ and let β_1 and β_2 be two functions of the set $\mathcal{F}(C, B)$ such that $\gamma\big((\alpha_1, \beta_1)\big) = \gamma\big((\alpha_2, \beta_2)\big)$. It follows that $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$ implying that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$.

Step 2. *The function* $\gamma : \mathcal{F}(C, A) \times \mathcal{F}(C, B) \to \mathcal{F}(C, A \times B)$ *is surjective:*

For, let $\delta : C \to A \times B$ be an element of the set $\mathcal{F}(C, A \times B)$. Let $\mathrm{pr}_A : A \times B \to A$, $\mathrm{pr}_A : (a, b) \mapsto a$ and $\mathrm{pr}_B : A \times B \to B$, $\mathrm{pr}_B : (a, b) \mapsto b$ be the projections of the set $A \times B$ onto the sets A and B, respectively.

Then the functions $\delta_A := \mathrm{pr}_A \circ \delta : C \to A$ and $\delta_B := \mathrm{pr}_B \circ \delta : C \to B$ are contained in the sets $\mathcal{F}(C, A)$ and $\mathcal{F}(C, B)$, respectively, and we have $\gamma(\delta_A, \delta_B) = \delta$.

Step 3. We have $(\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}}$:

It follows from Step 1 and Step 2 that

$$(a \cdot b)^c = |\mathcal{F}(C, A \times B)| = |\mathcal{F}(C, A) \times \mathcal{F}(C, B)| = |\mathcal{F}(C, A)| \cdot |\mathcal{F}(C, B)| = a^c \cdot b^c.$$

(c) For a function $\alpha : B \times C \to A$ and for an element c of the set C, let

$$\alpha_c : B \to A \text{ be defined by } \alpha_c : b \to \alpha(b, c).$$

Define the function

$$\gamma : \mathcal{F}(B \times C, A) \to \mathcal{F}(C, \mathcal{F}(B, A)) \text{ by } \gamma : \alpha \mapsto \big(c \mapsto \alpha_c\big).$$

Step 1. *The function $\mathcal{F}(B \times C, A) \to \mathcal{F}(C, \mathcal{F}(B, A))$ is injective:*

For, let α and β be two functions from the set $\mathcal{F}(B \times C, A)$ such that $\gamma(\alpha) = \gamma(\beta)$. It follows that $\alpha_c(b) = \beta_c(b)$ for all elements b and c of the sets B and C, respectively. Hence, we have

$$\alpha(b, c) = \alpha_c(b) = \beta_c(b) = \beta(b, c) \text{ for all } b \in B \text{ and } c \in C$$

implying that $\alpha = \beta$.

Step 2. *The function $\mathcal{F}(B \times C, A) \to \mathcal{F}(C, \mathcal{F}(B, A))$ is surjective:*

For, let $\alpha : C \to \mathcal{F}(B, A))$ with $c \mapsto \alpha_c$ be an element of the set $\mathcal{F}(C, \mathcal{F}(B, A))$. For each element b of the set B and each element c of the set C, set $\beta(b, c) := \alpha_c(b)$. Hence, we obtain a function $\beta : B \times C \to A$ such that $\gamma(\beta) = \alpha$.

Step 3. *We have $(a \cdot b)^c = a^c \cdot b^c$:*

It follows from Step 1 and Step 2 that

$$a^{bc} = |\mathcal{F}(B \times C, A)| = |\mathcal{F}(C, \mathcal{F}(B, A))| = \left(a^b\right)^c.$$

$\square$

6.9 Theorem. *Let a, b and c be three cardinal numbers. If $a \leqslant b$, then we have $a^c \leqslant b^c$.*

Proof. Since $a \leqslant b$, it follows from Proposition 2.5 that there exist two sets A and B such that $|A| = a$, $|B| = b$ and $A \subseteq B$. It follows that we have $\mathcal{F}(C, A) \subseteq \mathcal{F}(C, B)$. Again, by Proposition 2.5, it follows that

$$a^c = |\mathcal{F}(C, A)| \leqslant |\mathcal{F}(C, B)| = b^c.$$

$\square$

The Cardinality of the Power Set of a Set:

6.10 Theorem. *Let A be a set, let $\mathcal{P}(A)$ be the power set of the set A, and let $\mathcal{F}\big(A, \{0, 1\}\big)$ be the set of the functions from the set A into the set $\{0, 1\}$.*

Then the set $\mathcal{P}(A)$ and the set $\mathcal{F}\big(A, \{0, 1\}\big)$ have the same cardinality.

Proof. We define the mapping $\alpha : \mathcal{P}(A) \to \mathcal{F}\big(A, \{0, 1\}\big)$ from the power set $\mathcal{P}(A)$ of the set A into the set $\mathcal{F}\big(A, \{0, 1\}\big)$ by $\alpha : X \mapsto f_X$ where the function $f_X : A \to \{0, 1\}$ is defined by

$$f_X : x \mapsto \begin{cases} 0 & \text{if} \quad x \notin X \\ 1 & \text{if} \quad x \in X. \end{cases}.$$

One easily verifies that the function $\alpha : \mathcal{P}(A) \to \mathcal{F}\big(A, \{0, 1\}\big)$ is bijective. $\square$

6.11 Theorem. *Let A be a set of cardinality $\mathfrak{a}$.*

(a) We have

$$|\mathcal{P}(A)| = 2^{\mathfrak{a}}.$$

(b) We have $\mathfrak{a} < 2^{\mathfrak{a}}$ for all cardinal numbers $\mathfrak{a}$.

Proof. (a) The assertion follows from Theorem 6.10 and Definition 6.1.

(b) The assertion follows from (a) and the Theorem of Cantor (Theorem 2.6). □

Historical Notes:

The idea to define the power $\mathfrak{a}^{\mathfrak{b}}$ of two cardinal numbers $\mathfrak{a} = |A|$ and $\mathfrak{b} = |B|$ by the formula

$$\mathfrak{a}^{\mathfrak{b}} = |\mathcal{F}(B, A)|$$

(see Definition 6.1) is due to Georg Cantor:

Die Gesamtheit aller verschiedenen Belegungen von N mit M bildet eine bestimmte Menge mit den Elementen $f(N)$; wir nennen sie die **Belegungsmenge von N mit M** *und bezeichnen sie durch $(N \mid M)$. Es ist also*

$$(N \mid M) = \{f(N)\}.$$

Die Kardinalzahl von $(N \mid M)$ [...] dient uns zur Definition der Potenz $\mathfrak{a}^{\mathfrak{b}}$:

$$\mathfrak{a}^{\mathfrak{b}} = \overline{\overline{(N \mid M)}}.$$

See [Cantor 1895, p. 487].

The entirety of all different assignments of N with M forms a specific set with the elements $f(N)$; we call it the **set of assignments of N with M** *and denote it by $(N \mid M)$. Hence, we have*

$$(N \mid M) = \{f(N)\}.$$

The cardinal number of $(N \mid M)$ [...] serves us to define the power $\mathfrak{a}^{\mathfrak{b}}$:

$$\mathfrak{a}^{\mathfrak{b}} = \overline{\overline{(N \mid M)}}.$$

An assignment of N with M is just a function $f : N \to M$ from the set N into the set M. So we have $(N \mid M) = \mathcal{F}(N, M)$. Hence, the equation $\mathfrak{a}^{\mathfrak{b}} = \overline{\overline{(N \mid M)}}$ means $\mathfrak{a}^{\mathfrak{b}} = |\mathcal{F}(N, M)|$.

As a next step Cantor proves the equations $\mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b}+\mathfrak{c}}$, $\mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}} = (\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}}$ and $(\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} = \mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{c}}$ (Theorem 6.8):

[...] aus denen [...] die für drei beliebige Kardinalzahlen $\mathfrak{a}$, $\mathfrak{b}$ und $\mathfrak{c}$ gültigen Sätze sich ergeben:

$$\begin{aligned} \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}} &= \mathfrak{a}^{\mathfrak{b}+\mathfrak{c}}, \\ \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}} &= (\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}}, \\ (\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} &= \mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{c}}. \end{aligned}$$

See [Cantor 1895, p. 487].

[...] which imply [...] for each three arbitrary cardinal numbers $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ the following valid theorems:

$$\begin{aligned} \mathfrak{a}^{\mathfrak{b}} \cdot \mathfrak{a}^{\mathfrak{c}} &= \mathfrak{a}^{\mathfrak{b}+\mathfrak{c}}, \\ \mathfrak{a}^{\mathfrak{c}} \cdot \mathfrak{b}^{\mathfrak{c}} &= (\mathfrak{a} \cdot \mathfrak{b})^{\mathfrak{c}}, \\ (\mathfrak{a}^{\mathfrak{b}})^{\mathfrak{c}} &= \mathfrak{a}^{\mathfrak{b} \cdot \mathfrak{c}}. \end{aligned}$$

7 Countable Sets

Definition and Elementary Properties of Countable Sets:

In this section we will investigate countable sets (a set A is called **countable** if we have $|A| \leqslant \aleph_0$ (Definition 7.1)). Most results of this section prepare more general results to be explained in Section 8.

The most important results of this section are that that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ and that $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ if $\mathfrak{a} = |A|$ for a countably infinite set A (Theorem 7.8 and 7.6), that the countable unit of countable sets is countable (Theorem 7.11) and that

$$|A| = \big|\{X \subseteq A \mid X \text{ finite}\}\big|$$

for all countably infinite sets A (Theorem 7.12).

7.1 Definition. (a) A set A is called a **countable set** if we have $|A| \leqslant \aleph_0$.

(b) A set A is called **countably infinite** if we have $|A| = \aleph_0$.

Countable sets have been introduced in Unit *Cardinal Numbers* [Garden 2020i].

7.2 Theorem. *Let A be an infinite set. Then the set A has a countably infinite subset.*

Proof. Since the set A is infinite, it follows from Theorem 2.8 that we have $\aleph_0 \leqslant |A|$. By Theorem 2.4, there exists an injective mapping $\alpha : \mathbb{N}_0 \to A$ from the set $\mathbb{N}_0$ into the set A. It follows that the set

$$\{\alpha(n) \mid n \in \mathbb{N}_0\}$$

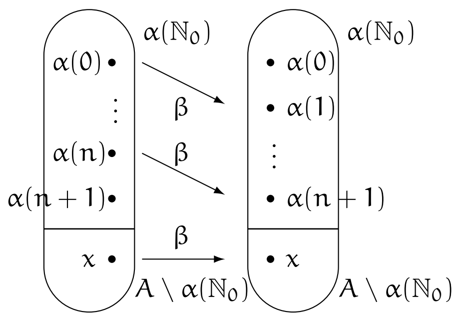
is a countably infinite subset of the set A. □

7.3 Theorem. *Let $\mathfrak{a}$ be an infinite cardinal number and let n be a natural number. Then we have $\mathfrak{a} + n = \mathfrak{a}$.*

Proof. Let A be a set such that $|A| = \mathfrak{a}$.

Step 1. *We have $\mathfrak{a} + 1 = \mathfrak{a}$:*

It follows from $|A| = \infty$ that $\aleph_0 \leqslant |A|$ (Proposition 2.8). By Theorem 2.4, there exists an injective function $\alpha : \mathbb{N}_0 \to A$ from the set $\mathbb{N}_0$ into the set A. Define the function $\beta : A \to A \setminus \{\alpha(0)\}$ by



$$\beta(x) := \begin{cases} \alpha(n+1) & \text{if} \quad x = \alpha(n) \text{ for an element } n \in \mathbb{N}_0 \\ x & \text{if} \quad x \notin \alpha(\mathbb{N}_0) \end{cases}$$

Since the function $\beta : A \to A \setminus \{\alpha(0)\}$ is bijective, it follows that

$$a = |A| = |A \setminus \{\alpha(0)\} \cup \{\alpha(0)\}| = |A \setminus \{\alpha(0)\}| + |\{\alpha(0)\}| = |A| + 1 = a + 1.$$

Step 2. *We have $a + n = a$ for all natural numbers n:*

We proceed by induction on n:

$n = 0$: By Proposition 4.5, we have $a + 0 = a$.

$n \mapsto n+1$: By induction and by Step 1, we have $a + n + 1 = (a + n) + 1 = a + 1 = a$. □

Direct Products of Countable Sets:

7.4 Theorem. (Cantor) *Let the mapping $\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ be defined by*

$$\alpha(m, n) := m + \sum_{j=0}^{m+n} j.$$

The mapping $\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is bijective.

Proof. Step 1. *The mapping $\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is injective:*

Suppose that we have $\alpha(m_1, n_1) = \alpha(m_2, n_2)$ for some natural numbers m_1, n_1, m_2 and n_2. It follows that

$$m_1 + \sum_{j=0}^{m_1+n_1} j = m_2 + \sum_{j=0}^{m_2+n_2} j. \tag{1}$$

Case 1. *Suppose that we have $m_1 + n_1 = m_2 + n_2$.*

It follows from Equation (1) that $m_1 = m_2$. Since $m_1 + n_1 = m_2 + n_2$, we also have $n_1 = n_2$.

Case 2. *Suppose w.l.o.g. that we have $m_1 + n_1 < m_2 + n_2$.*

It follows from Equation (1) that

$$m_1 = m_2 + \sum_{j=m_1+n_1+1}^{m_2+n_2} j \geqslant m_2 + m_1 + n_1 + 1 > m_1,$$

a contradiction.

Step 2. *The mapping $\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0$ is surjective:*

Let r be a natural number. Let a be the natural number such that

$$b := \sum_{j=0}^{a} j \leqslant r \text{ and } \sum_{j=0}^{a+1} j > r.$$

Since $b \leqslant r$, there exists a natural number m such that $m + b = r$. We claim that $m \leqslant a$: Otherwise, we have $m \geqslant a + 1$ implying that

$$r = b + m = \left(\sum_{j=0}^{a} j\right) + m \geqslant \left(\sum_{j=0}^{a} j\right) + a + 1 = \sum_{j=0}^{a+1} j > r,$$

a contradiction.

It follows from $m \leqslant a$ that there exists a natural number n such that $a = m + n$. Altogether, we have

$$\alpha(m,n) = m + \sum_{j=0}^{m+n} j = m + \sum_{j=0}^{a} j = m + b = r.$$

□

7.5 Remarks. (a) Note that we have

$$\sum_{j=0}^{n} j = \frac{n(n+1)}{2} \quad \text{for all } n \in \mathbb{N}_0.$$

Hence, the mapping $\alpha : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0$ from Theorem 7.4 becomes

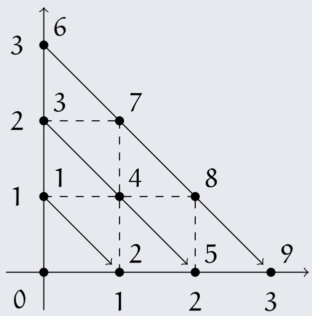
$$\alpha(m,n) = m + \frac{(m+n)(m+n+1)}{2}$$

which is a polynomial of degree 2 in the variables m and n.

(b) Let $\beta := \alpha^{-1} : \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$. Then we have

$$\beta(0) = (0,0),\ \beta(1) = (0,1),\ \beta(2) = (1,0),\ \beta(3) = (0,2),\ \beta(4) = (1,1),\ \beta(5) = (2,0)$$

and so on. The mapping $\beta : \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0$ may be visualized as follows:



7.6 Theorem. *(a) We have $|\mathbb{N}_0 \times \mathbb{N}_0| = |\mathbb{N}_0| \cdot |\mathbb{N}_0| = |\mathbb{N}_0|$.*

(b) Let A and B be two countable sets. Then the direct product $A \times B$ is also countable. If the sets A and B are countably infinite, we have $|A \times B| = |A| = |B|$.

(c) Let A be a countably infinite set, and let $\mathfrak{a} := |A|$. Then we have $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$.

Proof. The assertions follow directly from Theorem 7.4. □

7.7 Remark. We will see in Theorem 8.7 that we even have $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ for all infinite cardinal numbers $\mathfrak{a}$.

Unions of Countable Sets:

7.8 Theorem. *(a) Let A and B be two countable sets. Then the set $A \cup B$ is also countable.*

(b) Let A be a countably infinite set, and let $\mathfrak{a} := |A|$. Then we have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.

Proof. (a) Since the sets A and B are countable, we have $|A| \leqslant |\mathbb{N}_0|$ and $|B| \leqslant |\mathbb{N}_0|$. It follows that

$$|A \cup B| \leqslant |A| + |B| \leqslant |\mathbb{N}_0| + |\mathbb{N}_0| = 2 \cdot |\mathbb{N}_0| \leqslant |\mathbb{N}_0| \cdot |\mathbb{N}_0| = |\mathbb{N}_0|.$$

(b) We have

$$|\mathbb{N}_0| = |A| = \mathfrak{a} \leqslant \mathfrak{a} + \mathfrak{a} = |A| + |A| = |\mathbb{N}_0| + |\mathbb{N}_0| = 2 \cdot |\mathbb{N}_0| \leqslant |\mathbb{N}_0| \cdot |\mathbb{N}_0| = |\mathbb{N}_0|,$$

hence $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$. □

7.9 Remark. We will see in Theorem 8.3 that we even have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ for all infinite cardinal numbers $\mathfrak{a}$.

The Set of the Finite Subsets of a Countable Set:

7.10 Theorem. *Let $A_1, \dots, A_n$ be a finite family of countable sets for some natural number n. Then the set $\bigcup_{i=1}^n A_i$ is also countable.*

Proof. We proceed by induction on n:

$n = 1$: By assumption, the set

$$\bigcup_{i=1}^{1} A_i = A_1$$

is countable.

$n \mapsto n+1$: Suppose that the union $\bigcup_{i=1}^n A_i$ is countable. It follows from Theorem 7.8 that the set

$$\bigcup_{i=1}^{n+1} A_i = \Big(\bigcup_{i=1}^{n} A_i\Big) \cup A_{n+1}$$

is countable. □

7.11 Theorem. *Let $(A_i)_{i \in I}$ be a countable family of countable sets (that is, the set I is countable). Then the set $\bigcup_{i \in I} A_i$ is also countable.*

Proof. Since the sets A_i are countable for each element i of the set I and since the set I is countable, we have

$$|A_i| \leqslant |\mathbb{N}_0| \text{ for all } i \in I \text{ and } |I| \leqslant |\mathbb{N}_0|.$$

It follows that

$$\Big|\bigcup_{i \in I} A_i\Big| \leqslant \sum_{i \in I} |A_i| \leqslant \sum_{i \in I} |\mathbb{N}_0| = |I| \cdot |\mathbb{N}_0| \leqslant |\mathbb{N}_0| \cdot |\mathbb{N}_0| = |\mathbb{N}_0|.$$

□

7.12 Theorem. *Let A be a countably infinite set. Then the set*

$$\mathcal{C} := \{X \subseteq A \mid X \text{ is finite}\}$$

of the finite subsets of the set A is countably infinite.

Proof. Step 1. *Let n be a natural number, and let $\mathcal{C}_n := \{X \subseteq A \mid |X| \leqslant n\}$. Then the set $\mathcal{C}_n$ is countable:*

We proceed by induction on n:

$n = 0$: We have $\mathcal{C}_0 = \{\emptyset\}$ implying that the set $\mathcal{C}_0$ is countable.

$n \mapsto n+1$: We have

$$\mathcal{C}_{n+1} = \bigcup_{x \in A} \{X \cup \{x\} \mid X \in \mathcal{C}_n\} \cup \{\emptyset\}.$$

Let x be an element of the set A. Since the set $\mathcal{C}_n$ is countable (by induction), the set $\{X \cup \{x\} \mid X \in \mathcal{C}_n\}$ is also countable. Since the set A is countable, it follows from Theorem 7.11 that the set $\mathcal{C}_{n+1}$ is also countable.

Step 2. *The set $\mathcal{C}$ is countable:*

The assertion follows from Theorem 7.11 and the fact that $\mathcal{C} = \bigcup_{n \in \mathbb{N}_0} \mathcal{C}_n$. □

7.13 Remark. We will see in Theorem 8.11 that we even have

$$|A| = \big|\{X \subseteq A \mid X \text{ finite}\}\big|$$

for all infinite sets A.

Historical Notes:

The result that any infinite set has a countable subset (see Theorem 7.2) is due to Georg Cantor:

A. *Jede transfinite Menge T hat Teilmengen mit der Kardinalzahl $\aleph_0$.*

See [Cantor 1895, p. 493].

A. *Every transfinite set T has subsets with cardinal number $\aleph_0$.*

Transfinite means infinite. $\aleph_0$ (speak aleph 0) is another notation for the cardinal number $\mathbb{N}_0$.

Theorem 7.4 is also due to Cantor:

Es hat nämlich die Funktion

$$\mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2},$$

wie leicht zu zeigen, die bemerkenswerte Eigenschaft, dass sie alle positiven ganzen Zahlen und jede nur einmal darstellt, wenn in ihr μ und ν unabhängig voneinander ebenfalls jeden positiven, ganzzahligen Wert erhalten.

See [Cantor 1878, p. 257.]

As it is easy to show, the function

$$\mu + \frac{(\mu + \nu - 1)(\mu + \nu - 2)}{2}$$

has the remarkable property that it represents each positive integer once and only once if μ and ν run independently through all positive integers.

8 Arithmetic of Infinite Cardinal Numbers

In this section we will explain the main results of this unit, namely that $\mathfrak{a}+\mathfrak{a}=\mathfrak{a}$ and $\mathfrak{a}\cdot\mathfrak{a}=\mathfrak{a}$ for all infinite cardinal numbers (Theorem 8.3 and 8.7).

Together with Theorem 6.11 ($|\mathcal{P}(A)| = 2^{\mathfrak{a}}$ and $\mathfrak{a} < 2^{\mathfrak{a}}$ for all sets A with $|A| = \mathfrak{a}$) we are able to answer the question about the realtion between cardinalities and operations on sets in the following way:

We have $|A \cup B| = \max\{|A|, |B|\}$ if at least one of the two sets A and B is infinite (Corollary 8.5).

We have $|A \times B| = \max\{|A|, |B|\}$ if at least one of the two sets A and B is infinite (Theorem 8.9).

There are no general rules for the computation of the cardinality of the sets $A \cap B$ or $A \setminus B$. We just have

$$|A \cap B| \leqslant \min\{|A|, |B|\} \text{ and } |A \setminus B| \leqslant |A|.$$

For each cardinal number $\mathfrak{r} \leqslant \min\{|A|, |B|\}$ and $\mathfrak{s} \leqslant |A|$ there exists a set B and a set C such that

$$\mathfrak{r} = |A \cap B| \text{ and } \mathfrak{s} = |A \setminus B| \text{ (Theorem 8.10).}$$

Finally, we will show that

$$|A| = \big|\{X \subseteq A \mid X \text{ is finite}\}\big|$$

for all infinite sets A (Theorem 8.11) and the Theorem of König and Zermelo saying that

$$\sum_{i \in I} \mathfrak{a}_i < \prod_{i \in I} \mathfrak{b}_i.$$

if $\mathfrak{a}_i < \mathfrak{b}_i$ for all elements i of an index set I (Theorem 8.12).

*

We will need the Lemma of Zorn (Theorem 8.1) and Proposition 8.2 about direct products in the proof of Theorem 8.3 saying that $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$ for all infinite cardinal numbers $\mathfrak{a}$:

8.1 Theorem. (Lemma of Zorn) *Let $\mathbb{A} = (A, \leqslant)$ be a (partially) ordered set such that every chain C of the set A has an upper bound in the set A. (A chain of the set A is a totally ordered subset of the set A.)*

Then the set A contains a maximal element.

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

8.2 Proposition. *Let I and J be two index sets, and let $(A_i)_{i \in I}$ and $(B_j)_{j \in J}$ be two families of sets.*

(a) We have

$$\Big(\bigcup_{i \in I} A_i\Big) \times \Big(\bigcup_{j \in J} B_j\Big) = \bigcup_{(i,j) \in I \times J} (A_i \times B_j).$$

(b) We have

$$\Big(\bigcap_{i \in I} A_i\Big) \times \Big(\bigcap_{j \in J} B_j\Big) = \bigcap_{(i,j) \in I \times J} (A_i \times B_j).$$

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020c]. □

8.3 Theorem. *Let $\mathfrak{a}$ be an infinite cardinal number. Then we have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.*

Proof. Let A be a set such that $|A| = \mathfrak{a}$.

Step 1. *Definition of the set $\mathcal{F}$:*

Let $\mathcal{F} := \{\alpha : X \times \{0, 1\} \to X \mid X \subseteq A \text{ and } \alpha \text{ bijective}\}$.

Step 2. *The set $\mathcal{F}$ is non-empty:*

By Theorem 7.2, there exists a countably infinite subset C of the set A. By Theorem 7.4, we have $|C| \cdot 2 = |C|$, that is, there exists a bijective function $\alpha : C \times \{0, 1\} \to C$ from the set $C \times \{0, 1\}$ onto the set C.

Step 3. *Definition of a partial order on the set $\mathcal{F}$:*

Let X and Y be two subsets of the set A, and let $\alpha : X \times \{0, 1\} \to X$ and $\beta : Y \times \{0, 1\} \to Y$ be two bijective functions from the sets $X \times \{0, 1\}$ and $Y \times \{0, 1\}$ onto the sets X and Y, respectively. We set

$$\alpha \leqslant \beta \text{ if and only if } X \subseteq Y \text{ and } \alpha = \beta|_{X \times \{0,1\}}.$$

Step 4. *Every chain of the set $\mathcal{F}$ has an upper bound:*

Let I be a totally ordered index set, and $(\alpha_i)_{i \in I}$ be a chain in the set $\mathcal{F}$. Then for each element i of the set I, there exists a subset X_i of the set A and a bijective function $\alpha_i : X_i \times \{0, 1\} \to X_i$ with the following properties:

We have $X_i \subseteq X_j$ for all elements i and j of the set I such that $i \leqslant j$.

The function $\alpha_i : X_i \times \{0, 1\} \to X_i$ is induced by the function $\alpha_j : X_j \times \{0, 1\} \to X_j$ whenever $i \leqslant j$.

Let $X := \bigcup_{i \in I} X_i$. By Proposition 8.2, we have

$$X \times \{0, 1\} = \Big(\bigcup_{i \in I} X_i\Big) \times \{0, 1\} = \bigcup_{i \in I} \big(X_i \times \{0, 1\}\big).$$

By Proposition 3.3, there exists a bijective function $\alpha : X \times \{0, 1\} \to X$ such that $\alpha|_{X_i \times \{0,1\}} = \alpha_i$ for all elements i of the set I. It follows that the function α is an element of the set $\mathcal{F}$ such that $\alpha_i \leqslant \alpha$ for all elements i of the set I, that is, the function α is an upper bound of the chain $(\alpha_i)_{i \in I}$.

Step 5. *There exists a maximal element $\gamma : Z \times \{0, 1\} \to Z$ of the set $\mathcal{F}$*:

The assertion follows from Step 2, Step 4 and the Lemma of Zorn (Theorem 8.1).

Step 6. *The set $A \setminus Z$ is finite:*

Assume that the set $A \setminus Z$ is infinite. By Theorem 7.2, there exists a countably infinite set Y which is a subset of the set $A \setminus Z$. Since $|Y| \cdot 2 = |Y|$ (Theorem 7.4), there exists a bijective function $\delta : Y \times \{0, 1\} \to Y$. Since the set Y is a subset of the set $A \setminus Z$, we have $Y \cap Z = \emptyset$ implying that $(Y \times \{0, 1\}) \cap (Z \times \{0, 1\}) = \emptyset$. By Proposition 3.1, there exists a bijective function

$$\tau : (Y \times \{0, 1\}) \cup (Z \times \{0, 1\}) = (Y \cup Z) \times \{0, 1\} \to Y \cup Z$$

such that $\tau|_{Y \times \{0,1\}} = \delta$ and $\tau|_{Z \times \{0,1\}} = \gamma$. In particular, we have $\gamma < \tau$, in contradiction to the maximality of γ.

Step 7. *We have $\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$:*

Let Z be the subset of the set A defined in Step 5. Since the function $\gamma : Z \times \{0, 1\} \to Z$ is bijective, it follows from Proposition 5.6 that

$$|Z| + |Z| = |Z| \cdot 2 = |Z|.$$

Since the set $A \setminus Z$ is finite, it follows from Proposition 4.11 and Theorem 7.3 that

$$\mathfrak{a} = |A| = |A \setminus Z| + |Z| = |Z|.$$

Altogether, we get $\mathfrak{a} = |Z| = |Z| + |Z| = \mathfrak{a} + \mathfrak{a}$. □

8.4 Theorem. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers such that at least one of them is infinite. Then we have $\mathfrak{a} + \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$.*

Proof. Suppose that $\mathfrak{b}$ is infinite and that $\mathfrak{b} \geqslant \mathfrak{a}$. By Theorem 8.3, we have

$$\mathfrak{b} \leqslant \mathfrak{a} + \mathfrak{b} \leqslant \mathfrak{b} + \mathfrak{b} = \mathfrak{b}.$$

It follows that $\mathfrak{a} + \mathfrak{b} = \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$. □

8.5 Corollary. *Let A and B be two sets. If at least one of the two sets A and B is infinite, then we have*

$$|A \cup B| = |A| + |B| = \max\{|A|, |B|\}.$$

Proof. W.l.o.g. suppose that $|B| \geqslant |A|$. It follows from Proposition 4.10 and Theorem 8.4 that

$$|B| \leqslant |A \cup B| \leqslant |A| + |B| = \max\{|A|, |B|\} = |B|,$$

hence $|A \cup B| = |A| + |B| = \max\{|A|, |B|\}$. □

We need Proposition 8.6 about the direct product in the proof of Theorem 8.7 saying that $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ for all infinite cardinal numbers $\mathfrak{a}$.

8.6 Proposition. *Let I be a totally ordered index set, and let $(A_i)_{i \in I}$ be a family of sets such that the set A_j is a subset of the set A_k if $j \leqslant k$. Then we have*

$$\Big(\bigcup_{i \in I} A_i\Big) \times \Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} (A_i \times A_i).$$

Proof. See Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020d]. □

8.7 Theorem. *Let $\mathfrak{a}$ be an infinite cardinal number. Then we have $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$.*

Proof. Let A be a set such that $|A| = \mathfrak{a}$.

Step 1. *Definition of the set $\mathcal{F}$:*

Let $\mathcal{F} := \{\alpha : X \times X \to X \mid X \subseteq A \text{ and } \alpha \text{ bijective}\}$.

Step 2. *The set $\mathcal{F}$ is non-empty:*

By Theorem 7.2, there exists a countably infinite subset Z of the set A. By Theorem 7.4, there exists a bijective function $\alpha : Z \times Z \to Z$ from the set $Z \times Z$ onto the set Z.

Step 3. *Definition of a partial order on the set $\mathcal{F}$:*

Let X and Y be two subsets of the set A, and let $\alpha : X \times X \to X$ and $\beta : Y \times Y \to Y$ be two bijective functions from the sets $X \times X$ and $Y \times Y$ onto the sets X and Y, respectively. We set

$$\alpha \leqslant \beta \text{ if and only if } X \subseteq Y \text{ and } \alpha = \beta|_{X \times X}.$$

Step 4. *Every chain of the set $\mathcal{F}$ has an upper bound:*

Let I be an ordered index set, and $(\alpha_i)_{i \in I}$ be a chain in the set $\mathcal{F}$. Then for each element i of the set I, there exists a subset X_i of the set A such that the function $\alpha_i : X_i \times X_i \to X_i$ is bijective. Since the family $(\alpha_i)_{i \in I}$ is a chain, we have

$$X_i \subseteq X_j \text{ if } i \leqslant j \text{ for all } i, j \in I.$$

Let $X := \bigcup_{i \in I} X_i$. By Proposition 8.6, we have

$$X \times X = \Big(\bigcup_{i \in I} X_i\Big) \times \Big(\bigcup_{i \in I} X_i\Big) = \bigcup_{i \in I} (X_i \times X_i).$$

By Proposition 3.3, there exists a bijective function $\alpha : X \times X \to X$ such that $\alpha|_{X_i \times X_i} = \alpha_i$ for all elements i of the set I. It follows that the function α is an element of the set $\mathcal{F}$ such that $\alpha_i \leqslant \alpha$ for all elements i of the set I, that is, the function α is an upper bound of the chain $(\alpha_i)_{i \in I}$.

Step 5. *There exists a maximal element $\gamma : Z \times Z \to Z$ of the set $\mathcal{F}$:*

The assertion follows from Step 2, Step 4 and the Lemma of Zorn (Theorem 8.1).

Step 6. *We have $|Z| = |A|$:*

Since the set Z is a subset of the set A, we have $|Z| \leqslant |A|$. Assume that $|Z| < |A|$. Since

$$|A| = |A \setminus Z| + |Z| = \max\{|A \setminus Z|, |Z|\} \text{ (Theorem 8.4)},$$

it follows from $|Z| < |A|$ that $|A \setminus Z| = |A|$. In particular, we have $|Z| < |A \setminus Z|$.

It follows that there exists a subset Y of the set $A \setminus Z$ such that $|Y| = |Z|$. Note that we have $Y \cap Z = \emptyset$. It follows that

$$\begin{aligned}
|Z| \cdot |Z| &= |Z \times Z| = |Z| \\
|Y \times Z| &= |Y| \cdot |Z| = |Z| \cdot |Z| = |Z| \\
|Z \times Y| &= |Z| \cdot |Y| = |Z| \cdot |Z| = |Z| \\
|Y \times Y| &= |Y| \cdot |Y| = |Z| \cdot |Z| = |Z|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|(Z \times Y) \mathbin{\dot\cup} (Y \times Z) \mathbin{\dot\cup} (Y \times Y)| &= |Z \times Y| + |Y \times Z| + |Y \times Y| \\
&= |Z| + |Z| + |Z| = |Z| = |Y|.
\end{aligned}$$

Hence, there exists a bijective function

$$\beta : (Z \times Y) \mathbin{\dot\cup} (Y \times Z) \mathbin{\dot\cup} (Y \times Y) \to Y$$

from the set $(Z \times Y) \mathbin{\dot\cup} (Y \times Z) \mathbin{\dot\cup} (Y \times Y)$ onto the set Y.

Since the functions $\gamma : Z \times Z \to Z$ and $\beta : (Z \times Y) \mathbin{\dot\cup} (Y \times Z) \mathbin{\dot\cup} (Y \times Y) \to Y$ are bijective and since

$$(Z \times Z) \cap \big((Z \times Y) \mathbin{\dot\cup} (Y \times Z) \mathbin{\dot\cup} (Y \times Y)\big) = \emptyset \text{ and } Y \cap Z = \emptyset,$$

it follows from Proposition 3.1 that there exists a bijective function $\delta : (Z \cup Y) \times (Z \cup Y) \to Z \cup Y$ from the set $(Z \cup Y) \times (Z \cup Y)$ onto the set $Z \cup Y$. Since $Y \neq \emptyset$ (we have $|Y| = |Z|$), the function δ strictly extends the function γ and belongs to the set $\mathcal{F}$, a contradiction. Hence, we get $|Z| = |A|$.

Step 7. *We have* $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$*:*

Let Z be the subset of the set A defined in Step 5. It follows from Step 5 and Step 6 that

$$\mathfrak{a} = |A| = |Z| = |Z| \cdot |Z| = |A| \cdot |A| = \mathfrak{a} \cdot \mathfrak{a}.$$

□

8.8 Corollary. *Let* $\mathfrak{a}$ *be an infinite cardinal number. Then we have* $\mathfrak{a}^n = \mathfrak{a}$*.*

Proof. The assertion follows by induction from Theorem 8.7. □

8.9 Theorem. *(a) Let* $\mathfrak{a}$ *and* $\mathfrak{b}$ *be two cardinal numbers such that at least one of them is infinite, and suppose that* $\mathfrak{a} \neq 0$ *and* $\mathfrak{b} \neq 0$*. Then we have* $\mathfrak{a} \cdot \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$*.*

(b) Let A *and* B *be two non-empty sets, and suppose that at least one of them is of infinite cardinality. Then we have*

$$|A \times B| = \max\{|A|, |B|\}.$$

Proof. (a) Suppose that $\mathfrak{b}$ is infinite and that $\mathfrak{b} \geqslant \mathfrak{a}$. Since $\mathfrak{a} \geqslant 1$, it follows from Theorem 8.3 that we have

$$\mathfrak{b} = 1 \cdot \mathfrak{b} \leqslant \mathfrak{a} \cdot \mathfrak{b} \leqslant \mathfrak{b} \cdot \mathfrak{b} = \mathfrak{b}.$$

It follows that $\mathfrak{a} \cdot \mathfrak{b} = \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$.

(b) follows from (a). □

8.10 Theorem. *Let* A *and* B *be two sets.*

(a) We have

$$|A \cap B| \leqslant \min\{|A|, |B|\} \text{ and } |A \setminus B| \leqslant |A|.$$

(b) Let A *be a set. For each two cardinal numbers* $\mathfrak{r} \leqslant |A|$ *and* $\mathfrak{s} \leqslant |A|$ *there exist a set* B *and a set* C *such that*

$$\mathfrak{r} = |A \cap B| \text{ and } \mathfrak{s} = |A \setminus C|.$$

Proof. (a) Since the set $A \cap B$ is a subset of the sets A and B, it follows from Proposition 2.5 that

$$|A \cap B| \leqslant |A| \text{ and } |A \cap B| \leqslant |B|.$$

In the same way it follows that $|A \setminus B| \leqslant |A|$.

(b) Let $\mathfrak{r}$ be a cardinal number such that $\mathfrak{r} \leqslant |A|$, and let R be a set of cardinality $\mathfrak{r}$ (for example $R := \mathfrak{r}$). By Theorem 2.4, there exists an injective mapping $\alpha : R \to A$. Let $B := \alpha(R)$. It follows that $|B| = |\alpha(R)| = \mathfrak{r}$ and $A \cap B = B$.

Let $\mathfrak{s}$ be a cardinal number such that $\mathfrak{s} \leqslant |A|$. In the same way one may show that there exists a subset S of the set A of cardinality $\mathfrak{s}$. Set $C := A \setminus S$. It follows that $|A \setminus C| = |S| = \mathfrak{s}$. □

8.11 Theorem. *Let A be an infinite set. Then we have*

$$|A| = \big|\{X \subseteq A \mid X \textit{ is finite}\}\big|.$$

Proof. Let $\mathcal{F} := \{X \subseteq A \mid X \text{ finite}\}$. We have to show that $|A| = |\mathcal{F}|$.

Step 1. *We have* $|A| \leqslant |\mathcal{F}|$:

For, let $\alpha : A \to \mathcal{F}$ be the mapping from the set A into the set $\mathcal{F}$ defined by $\alpha : x \mapsto \{x\}$. The mapping $\alpha : A \to \mathcal{F}$ is obviously injective. It follows from Theorem 2.4 that $|A| \leqslant |\mathcal{F}|$.

Step 2. *We have* $|\mathcal{F}| \leqslant |A|$:

For each natural number n, let

$$\mathcal{F}_n := \{X \subseteq A \mid |X| = n\} = \{X \in \mathcal{F} \mid |X| = n\}.$$

Obviously, we have

$$\mathcal{F} = \bigcup_{n=0}^{\infty} \mathcal{F}_n.$$

For each set X of the set $\mathcal{F}_n$ there exists a bijective mapping $\beta_X : \{1, \ldots, n\} \to X$. We claim that the mapping $\beta : \mathcal{F}_n \to A^n$ defined by $\beta : X \mapsto \big(\beta_X(1), \ldots, \beta_X(n)\big)$ is injective:

For, let X and Y be two sets of the set $\mathcal{F}_n$ such that $\beta(X) = \beta(Y)$. It follows that

$$\beta_X(i) = \beta_Y(i) \text{ for all } i = 1, \ldots, n.$$

Hence, we get

$$X = \big\{\beta_X(1), \ldots, \beta_X(n)\big\} = \big\{\beta_Y(1), \ldots, \beta_Y(n)\big\} = Y.$$

It follows from Theorem 2.4 and from Corollary 8.8 that

$$|\mathcal{F}_n| \leqslant \big|A^n\big| = |A|^n = |A|.$$

It follows from Proposition 4.10, Proposition 5.6, Theorem 5.10 and Corollary 8.8 that

$$|\mathcal{F}| = \Big|\bigcup_{n=0}^{\infty} \mathcal{F}_n\Big| \overset{4.10}{\leqslant} \sum_{n=0}^{\infty} |\mathcal{F}_n| \leqslant \sum_{n=0}^{\infty} |A| \overset{5.6}{=} |\mathbb{N}_0| \cdot |A| \overset{5.10}{\leqslant} |A| \cdot |A| \overset{8.8}{=} |A|.$$

□

The Theorem of König and Zermelo:

8.12 Theorem. (König and Zermelo) *Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be two families of cardinal numbers, and suppose that we have*

$$a_i < b_i \textit{ for all } i \in I.$$

Then we have

$$\sum_{i \in I} a_i < \prod_{i \in I} b_i.$$

Proof. By Proposition 2.5, there exist two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ of sets such that

$$|A_i| = a_i,\ |B_i| = b_i \text{ and } A_i \subseteq B_i \text{ for all } i \in I.$$

Assume that we have

$$\sum_{i\in I} a_i \geqslant \prod_{i\in I} b_i.$$

Let

$$S := \bigcup_{i\in I} (A_i \times \{i\}).$$

Since

$$|S| = \sum_{i\in I} a_i \text{ and } \Big|\prod_{i\in I} B_i\Big| = \prod_{i\in I} b_i,$$

it follows from Theorem 2.3 that there exists a surjective mapping $\alpha : S \to \prod_{i\in I} B_i$ from the set S onto the set $\prod_{i\in I} B_i$.

Let i be an element of the set I. Note that for each element x of the set S the element $\alpha(x)$ is an element of the direct product $\prod_{i\in I} B_i$, that is, we have

$$\alpha(x) = \big(\alpha(x)_i\big)_{i\in I}.$$

Since

$$\big|\{\alpha(x)_i \mid x \in A_i \times \{i\}\}\big| \leqslant \big|A_i \times \{i\}\big| = \big|A_i\big| < \big|B_i\big|,$$

the set $\{\alpha(x)_i \mid x \in A_i \times \{i\}\}$ is a proper subset of the set B_i. Hence, for each element i of the set I there exists an element b_i of the set B_i not contained in the set $\{\alpha(x)_i \mid x \in A_i \times \{i\}\}$, that is,

$$b_i \neq \alpha(x)_i \text{ for all } x \in A_i \times \{i\}.$$

It follows that $b := (b_i)_{i\in I}$ is an element of the set $\prod_{i\in I} B_i$ such that

$$b \neq \alpha(x) \text{ for all } x \in \bigcup_{i\in I} (A_i \times \{i\}) = S,$$

in contradiction the the assumption that the mapping $\alpha : S \to \prod_{i\in I} B_i$ is surjective.

Note that the definition $b := (b_i)_{i\in I}$ makes use of the axiom of choice since we choose an element b of the set

$$\prod_{i\in I} \Big(B_i \setminus \big(\{\alpha(x)_i \mid x \in A_i \times \{i\}\}\big)\Big).$$

$\square$

Historical Notes:

The history of Theorem 8.7 ($a \cdot a = a$ for all infinite cardinal numbers a) is explained in detail in the paper *Der Multiplikationssatz der Mengenlehre* by Oliver Deiser [Deiser 2005]. Most of the following information stems from this excellent survey.

The first step in this direction is a theorem of Georg Cantor from 1878 where he shows that $|\mathbb{R}^n| \leqslant |\mathbb{R}|$:

(A.) *Sind* $x_1, \dots, x_n$ n *voneinander unabhängige, veränderliche reelle Größen, von denen jede alle Werte, die* $\geqslant 0$ *und* $\leqslant 1$ *sind, annehmen kann, und ist* t *eine andere Veränderliche mit dem gleichen Spielraum* ($0 \leqslant t \leqslant 1$), *so ist es möglich, die eine Größe* t *dem*

System der n *Größen* $x_1, \ldots, x_n$ *so zuzuordnen, dass zu jedem bestimmten Wert von* t *ein bestimmtes Wertsystem* $x_1, \ldots, x_n$ *und umgekehrt zu jedem bestimmten Wertsystem* $x_1, \ldots, x_n$ *ein gewisser Wert von* t *gehört.*

See [Cantor 1878, p. 245].

(A.) *Are* $x_1, \ldots, x_n$ n *mutually independent real variables, each of which can have all values that are* $\geqslant 0$ *and* $\leqslant 1$*, and is* t *another real variable in the same interval* ($0 \leqslant t \leqslant 1$)*, then it is possible to map the variable* t *to the system of* n *variables* $x_1, \ldots, x_n$ *so that every value of* t *corresponds to a system of values* $x_1, \ldots, x_n$ *and vice versa every system of values* $x_1, \ldots, x_n$ *corresponds to a value of* t.

In fact, the above theorem means that there exists a bijective mapping between the intervals $[0,1]^n$ and $[0,1]$. It is easy to see that the interval $[0,1]$ has the same cardinality as the set $\mathbb{R}$ of the real numbers. However, Dedekind pointed out that the proof of Cantor only shows that there is an injective mapping from the set $[0,1]^n$ into the set $[0,1]$, that is $|[0,1]^n| \leqslant |[0,1]|$.

Of course, the inequality $|[0,1]| \leqslant |[0,1]^n|$ is trivial, but in 1878 the Theorem of Cantor and Bernstein ($|A| \leqslant |B|$ and $|B| \leqslant |A|$ imply $|A| = |B|$, see Unit *Cardinal Numbers* [Garden 2020i]) was not yet known.

By the way, the result that $|\mathbb{R} \times \mathbb{R}| = |\mathbb{R}|$ was even for Cantor so surprising that he wrote in a letter to Dedekind: *je le vois, mais je ne crois pas* (*I see it, but I don't believe it*). The rest of the letter is in German.

In 1901 Felix Bernstein uses in his PhD thesis the multiplication theorem and refers to Cantor for a proof:

Hilfssatz 1. *Es sei* n *eine ganze endliche Zahl, so ist*

$$\aleph^n = \aleph$$

für jedes Aleph.

Den Beweis des Satzes, den ich aus mündlicher Mitteilung von G. Cantor kenne, führt man alalog wie im einfachsten Fall $\aleph = \aleph_0$ *durch Verwandlung einer Doppelreihe in eine einfache Reihe [...].*

See [Bernstein 1901] or [Bernstein 1905, p. 150]. [Bernstein 1901] is the PhD thesis of Bernstein which has been reprinted with minor modifications in the *Mathematische Annalen* [Bernstein 1905].

Lemma 1. *If* n *is a natural number, then we have*

$$\aleph^n = \aleph$$

for each Aleph.

The proof of the theorem, which I know from an oral communication by G. Cantor, is carried out in the same way as in the simplest case $\aleph = \aleph_0$ *by converting a double series into a single series [...].*

This is the only hint that Cantor was in possession of a proof of the general theorem $\mathfrak{a} \cdot \mathfrak{a} = \mathfrak{a}$ for all cardinal numbers $\mathfrak{a}$.

The first published proof stems from Philip Jourdain [Jourdain 1904], but the proof remains vague. It is A. E. Harward who gives the first proof in 1905:

11. *I now proceed to prove that* $\aleph_\beta^2 = \aleph_\beta$ *for all values of* β *(finite and transfinite).*

See [Harward 1905, p. 451].

In fact, the paper of Harward contains two proof ideas, the second in Note A of the paper.

Gerhard Hessenberg found a second proof of the multiplication theorem independently from Harward:

LXVII. *Ist $\aleph_\alpha$ höher oder gleich $\aleph_\beta$, so ist*

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\alpha.$$

Insbesondere ist $n \cdot \aleph_\alpha = \aleph_\alpha^n = \aleph_\alpha$ für endliches n.

See [Hessenberg 1906, p. 594].

LXVII. *If $\aleph_\alpha$ is greater or equal than $\aleph_\beta$, then we have*

$$\aleph_\alpha + \aleph_\beta = \aleph_\alpha \cdot \aleph_\beta = \aleph_\alpha.$$

In particular we have $n \cdot \aleph_\alpha = \aleph_\alpha^n = \aleph_\alpha$ for finite n.

In fact, Hessenberg provided a second proof of this theorem in [Hessenberg 1907]. Further proofs of the multiplication theorem stem from Philip Jourdain [Jourdain 1908, p. 506] and from Felix Hausdorff [Hausdorff 1914, p. 127]. Unfortunately, the proof of Harward of 1905 remained unnoticed. His proof has been more or less reinvented by Hausdorff in 1914. All these proofs use the result of Zermelo that every set can be well-ordered (for details see Unit *Well-Ordered Sets* [Garden 2020g]).

Finally, Max Zorn gave in [Zorn 1944] a proof based on his Lemma of Zorn. The proof of Theorem 8.7 explained above follows the ideas of Max Zorn.

I could not find out who gave the first proof of Theorem 8.11 ($|A| = |\{X \subseteq A \mid X \text{ is finite}\}|$ for each infinite set A). I have taken the proof of Bourbaki in [Bourbaki 2006]:

Proposition 5. *L'ensemble $\mathcal{F}(E)$ des parties finies d'un ensemble infini E est équipotent à E.*

See [Bourbaki 2006, E.III.50].

Proposition 5. *The set $\mathcal{F}(E)$ of the finite subsets of an infinite set E has the same cardinality as E.*

The Theorem of König and Zermelo (Theorem 8.12; $a_i < b_i$ for all $i \in I \Rightarrow \sum_{i \in I} a_i < \prod_{i \in I} b_i$) is due to Julius König for a special case (see [König 1905]) and to Ernst Zermelo for the general case (see [Zermelo 1908]):

33_{VI}. **Theorem.** *Sind zwei äquivalente Mengen T und T', deren Elemente $M, N, R, \ldots$ bzw. $M', N', R', \ldots$ unter sich elementfremde Mengen sind, so aufeinander abgebildet, dass jedes Element M von T von kleinerer Mächtigkeit ist als das entsprechende Element M' von T', so ist auch die Summe $S = \mathcal{S}T$ aller Elemente von T von kleinerer Mächtigkeit als das Produkt $P' = \mathcal{P}T'$ aller Elemente von T'.*

See [Zermelo 1908, p. 277].

33_{VI}. **Theorem.** *Let T und T' be two equivalent sets whose elements $M, N, R, \ldots$ and $M', N', R', \ldots$ are pairwise disjoint and suppose that each element M of T is of smaller cardinality than the corresponding element M' of T'. Then the sum $S = \mathcal{S}T$ of all elements of T is of smaller cardinality than the product $P' = \mathcal{P}T'$ of all elements of T'.*

9 Summary of the Unit Cardinal Arithmetic

Arithmetic of Cardinal Numbers:

9.1 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be three cardinal numbers.*

(a) We have $(\mathfrak{a}+\mathfrak{b})+\mathfrak{c}=\mathfrak{a}+(\mathfrak{b}+\mathfrak{c})$.

(b) We have $\mathfrak{a}+0=0+\mathfrak{a}=\mathfrak{a}$.

(c) We have $\mathfrak{a}+\mathfrak{b}=\mathfrak{b}+\mathfrak{a}$.

(d) We have $(\mathfrak{a}\cdot\mathfrak{b})\cdot\mathfrak{c}=\mathfrak{a}\cdot(\mathfrak{b}\cdot\mathfrak{c})$.

(e) We have $\mathfrak{a}\cdot 0=0\cdot\mathfrak{a}=0$ *and* $\mathfrak{a}\cdot 1=1\cdot\mathfrak{a}=\mathfrak{a}$.

(f) We have $\mathfrak{a}\cdot\mathfrak{b}=0$ *if and only if* $\mathfrak{a}=0$ *or* $\mathfrak{b}=0$.

(g) We have $\mathfrak{a}\cdot\mathfrak{b}=\mathfrak{b}\cdot\mathfrak{a}$.

(h) We have $\mathfrak{a}\cdot(\mathfrak{b}+\mathfrak{c})=\mathfrak{a}\cdot\mathfrak{b}+\mathfrak{a}\cdot\mathfrak{c}$ *and* $(\mathfrak{a}+\mathfrak{b})\cdot\mathfrak{c}=\mathfrak{a}\cdot\mathfrak{c}+\mathfrak{b}\cdot\mathfrak{c}$.

Proof. (a) and (c) follow from Theorem 4.4.

(b) follows from Proposition 4.5.

(d) and (g) follow from Theorem 5.7.

(e) and (f) follow from Proposition 5.5.

(h) follows from Theorem 5.9. □

9.2 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ and $\mathfrak{d}$ be four cardinal numbers.*

(a) If $\mathfrak{a}\leqslant\mathfrak{c}$ *and* $\mathfrak{b}\leqslant\mathfrak{d}$, *then we have* $\mathfrak{a}+\mathfrak{b}\leqslant\mathfrak{c}+\mathfrak{d}$.

(b) If $\mathfrak{a}\leqslant\mathfrak{c}$ *and* $\mathfrak{b}\leqslant\mathfrak{d}$, *then we have* $\mathfrak{a}\cdot\mathfrak{b}\leqslant\mathfrak{c}\cdot\mathfrak{d}$.

Proof. (a) follows from Theorem 4.6.

(b) follows from Theorem 5.10. □

9.3 Theorem. *Let $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$ be three cardinal numbers.*

(a) We have $\mathfrak{a}^0=1$ *and* $\mathfrak{a}^1=\mathfrak{a}$.

(b) We have $\mathfrak{a}^{\mathfrak{b}+\mathfrak{c}}=\mathfrak{a}^{\mathfrak{b}}\cdot\mathfrak{a}^{\mathfrak{c}}$.

(c) We have $(\mathfrak{a}\cdot\mathfrak{b})^{\mathfrak{c}}=\mathfrak{a}^{\mathfrak{c}}\cdot\mathfrak{b}^{\mathfrak{c}}$.

(d) We have $\left(\mathfrak{a}^{\mathfrak{b}}\right)^{\mathfrak{c}}=\mathfrak{a}^{\mathfrak{b}\mathfrak{c}}$.

(e) If $\mathfrak{a}\leqslant\mathfrak{b}$, *then we have* $\mathfrak{a}^{\mathfrak{c}}\leqslant\mathfrak{b}^{\mathfrak{c}}$.

Proof. (a) follows from Proposition 6.5.

(b), (c) and (d) follows from Theorem 6.8.

(e) follows from Theorem 6.9. □

9.4 Theorem. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two cardinal numbers, and suppose that $\mathfrak{a}$ is infinite.*

(a) We have $\mathfrak{a}+\mathfrak{a}=\mathfrak{a}$.

(b) We have $\mathfrak{a}+\mathfrak{b}=\max\{\mathfrak{a},\mathfrak{b}\}$.

(c) We have $\mathfrak{a}\cdot\mathfrak{a}=\mathfrak{a}$.

(d) If $\mathfrak{a} \neq 0$ *and* $\mathfrak{b} \neq 0$*, then we have* $\mathfrak{a} \cdot \mathfrak{b} = \max\{\mathfrak{a}, \mathfrak{b}\}$.

(e) We have $\mathfrak{a} < 2^{\mathfrak{a}}$.

Proof. (a) follows from Theorem 8.3.

(b) follows from Theorem 8.4.

(c) follows from Theorem 8.7.

(d) follows from Theorem 8.9.

(e) follows from Theorem 6.11. □

Cardinality of Sets:

9.5 Theorem. *Let* A *and* B *be two sets, and let* $(A_i)_{i \in I}$ *be a family of sets over an index set* I.

(a) We have $|A| = |A \setminus B| + |A \cap B|$.

(b) We have $|A| + |B| = |A \cup B| + |A \cap B|$.

(c) We have $|A| + |B| = |A \cup B| = \max\{|A|, |B|\}$ *if it at least one of the two sets* A *or* B *is infinite.*

(d) We have $|A \cup B| \leqslant |A| + |B|$.

(e) We have $\left|\bigcup_{i \in I} A_i\right| \leqslant \sum_{i \in I} |A_i|$.

(f) We have $|A \cap B| \leqslant \min\{|A|, |B|\}$ *for all sets* A *and* B*, and for each cardinal number* $0 \leqslant \mathfrak{c} \leqslant |A|$ *there exists a set* C *such that* $|A \cap C| = \mathfrak{c}$.

(g) We have $|A \setminus B| \leqslant |A|$ *for all sets* A *and* B*, and for each cardinal number* $0 \leqslant \mathfrak{c} \leqslant |A|$ *there exists a set* C *such that* $|A \setminus C| = \mathfrak{c}$.

Proof. (a) follows from Proposition 4.11.

(b) follows from Theorem 4.12.

(c) follows from Corollary 8.5.

(d) and (e) follow from Proposition 4.10.

(f) and (g) follow from Theorem 8.10. □

9.6 Theorem. *Let* A *be a set, and let* $\mathcal{P}(A)$ *be its power set. Then we have*

$$|\mathcal{P}(A)| = 2^{\mathfrak{a}}.$$

Proof. The assertion follows from Theorem 6.11. □

9.7 Theorem. *(a) Let* $(A_i)_{i \in I}$ *be a countable family of countable sets (that is, the set* I *is countable). Then the set* $\bigcup_{i \in I} A_i$ *is also countable.*

(b) Let A *be an infinite set. Then we have*

$$|A| = \left|\{X \subseteq A \mid X \text{ is finite}\}\right|.$$

Proof. (a) follows from Theorem 7.11.

(b) follows from Theorem 8.11. □

10 Notes and References

Do you want to learn more? The main subject of the walk *The Cardinality of Sets* is to explain the definition of the cardinality of sets and to explain the arithmetic of cardinal numbers. This has been explained in the present and the former units.

However, there is one more unit in this walk, namely the Unit *The Axiomatics of von Neumann, Bernays and Gödel*. The axiomatics of Zermelo and Fraenkel only knows one type of mathematical objects, namely the sets. This approach has the little disadvantage that there is no term for the collection of all sets since there is no set of all sets. The Axiomatics of von Neumann, Bernays and Gödel defines *two* types of mathematical objects, namely sets and classes. This allows us to speak of the class of all set or the class of all cardinal numbers. The details will be explained in Unit *The Axiomatics of von Neumann, Bernays and Gödel* [Garden 2020j].

11 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Bernstein, Felix (1901). "Untersuchungen aus der Mengenlehre: Inaugural-Dissertation". PhD thesis. Göttingen: Universität Göttingen (cit. on pp. 35, 39).

— (1905). "Untersuchungen aus der Mengenlehre". In: *Mathematische Annalen* 61.1, pp. 117–155. This article is the publication of the PhD-thesis of Felix Bernstein [Bernstein 1901] written in 1901; See DigiZeitschriften (cit. on p. 35).

Bourbaki, Nicolas (2006). *Théorie des ensembles*. Berlin, Heidelberg, and New York: Springer Verlag. This book is a reprint of the edition of 1970. The four chapters of this book have been first published separately in the years 1954 (Description de la mathématique formelle), 1939 and 1954 (Théorie des ensembles), 1956 (Ensembles ordonnés, cardinaux, nombres entiers) and 1956 (Structures). (Cit. on p. 36).

Cantor, Georg (1878). "Ein Beitrag zur Mannigfaltigkeitslehre". In: *Journal für die reine und angewandte Mathematik* 84, pp. 242–258. See DigiZeitschriften (cit. on pp. 27, 35).

— (1895). "Beiträge zur Begründung der transfiniten Mengenlehre 1". In: *Mathematische Annalen* 46, pp. 481–512. See DigiZeitschriften (cit. on pp. 12, 13, 16, 17, 22, 23, 27).

Deiser, Oliver (2005). "Der Multiplikationssatz der Mengenlehre". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 107.2, pp. 88–109. See DMV (cit. on p. 34).

Harward, A. E. (1905). "On Transfinite Numbers". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 6.10, pp. 439–460. See BHL (cit. on p. 36).

Hausdorff, Felix (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit and Comp. For a new edition see [Hausdorff 2002]. (Cit. on pp. 36, 39).

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Hessenberg, Gerhard (1906). *Grundbegriffe der Mengenlehre*. Göttingen: Vandenhoeck & Ruprecht (cit. on p. 36).

— (1907). "Potenzen transfiniter Ordnungszahlen". In: *Jahresbericht der Deutschen Mathematiker-Vereinigung* 16, pp. 130–137. See DigiZeitschrift (cit. on p. 36).

Jourdain, Philip E. B. (1904). "On the Transfinite Cardinal Numbers of Number-classes in general". In: *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* 6.7, pp. 294–303. See BHL (cit. on p. 35).
— (1908). "On the Multiplication of Alephs". In: *Mathematische Annalen* 65, pp. 506–512. See DigiZeitschriften (cit. on p. 36).
König, Julius (1905). "Zum Kontinuum-Problem". In: *Mathematische Annalen* 60, pp. 177–180. See DigiZeitschrift (cit. on p. 36).
Zermelo, Ernst (1908). "Untersuchungen über die Grundlagen der Mengenlehre I". In: *Mathematische Annalen* 65, pp. 261–281. See DigiZeitschriften (cit. on p. 36).
Zorn, Max (1944). "Idempotency of Infinite Cardinals". In: *University of California Publications* 2, pp. 9–12 (cit. on p. 36).

12 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *Direct Products and Relations*. URL: https://www.math-garden.com/unit/nst-direct-products (cit. on pp. 14, 15).
— (2020b). *Functions and Equivalent Sets*. URL: https://www.math-garden.com/unit/nst-functions (cit. on pp. 6, 19).
— (2020c). *Families and the Axiom of Choice*. URL: https://www.math-garden.com/unit/nst-families (cit. on pp. 7, 8, 20, 28).
— (2020d). *Ordered Sets and the Lemma of Zorn*. URL: https://www.math-garden.com/unit/nst-ordered-sets (cit. on pp. 8, 28, 30).
— (2020e). *The Natural Numbers and the Principle of Induction*. URL: https://www.math-garden.com/unit/nst-natural-numbers (cit. on pp. 8, 9, 13, 14, 18).
— (2020f). *Finite Sets and their Cardinalities*. URL: https://www.math-garden.com/unit/card-finite-sets (cit. on pp. 3, 5, 9, 14, 18).
— (2020g). *Well-Ordered Sets*. URL: https://www.math-garden.com/unit/card-well-ordered (cit. on pp. 3, 36).
— (2020h). *Ordinal Numbers*. URL: https://www.math-garden.com/unit/card-ordinal (cit. on p. 3).
— (2020i). *Cardinal Numbers*. URL: https://www.math-garden.com/unit/card-cardinal (cit. on pp. 3, 4, 6, 7, 23, 35).
— (2020j). *The Axiomatics of von Neumann, Bernays and Gödel*. In preparation (cit. on pp. 3, 39).

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