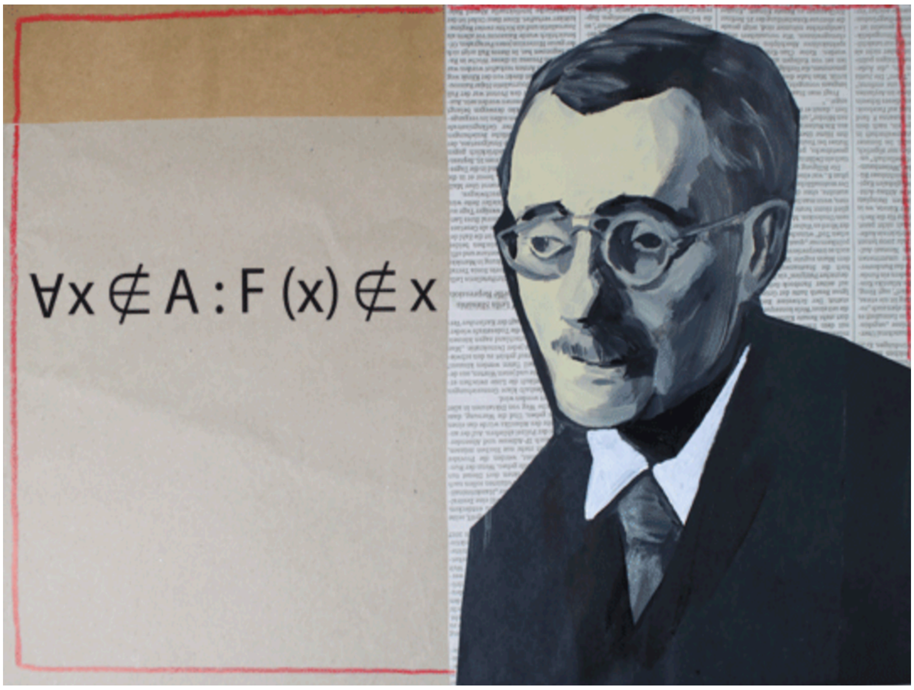
M. Garden

Families and The Axiom of Choice



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Title Page *Families and the Axiom of Choice*

Artwork by Gisela Naumann - www.malraum-bonn.de.

Version

Version 1.0.3 from October 2020

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Print

Printed with LaTeX.

Translations

For translations `https://www.leo.org`, `https://www.dict.cc` and `https://translate.google.com` have been used.

Contact

info@math-garden.com

Contents

1 Introduction

The present unit is part of the walk *The Axioms of Zermelo and Fraenkel*. It introduces into families of sets, the axiom of substitution and the axiom of choice.

Families and the Axiom of Substitution (see Section 2):

We often use expressions of the form *Let* $(A_i)_{i \in I}$ *be a family of sets ...* Intuitively, it is clear what the expression $(A_i)_{i \in I}$ means.

However, an axiomatic construction of mathematics brings the need to deduce each mathematical expression from the founding axioms. Since we use the axiomatic of Zermelo and Fraenkel (see Unit Universe [Garden 2020a]), we have to define the expression $(A_i)_{i \in I}$ as a set:

In the unit Functions [Garden 2020d] we have introduced the functions $f : A \to B$ from a set A into a set B as a specific set. Hence, we can define a family $(A_i)_{i \in I}$ as a function:

More precisely, an **index set** I is a non-empty set (see Definition 2.2). If $\mathcal{A}$ is a set (of sets), then a **family** is a function

$$f : I \to \mathcal{A}$$

from an index set I into the set $\mathcal{A}$. We set $A_i := f(i)$. The function $f : I \to \mathcal{A}$ is denoted by $(A_i)_{i \in I}$ (see Definition 2.3).

The advantage of this definition is that we have a formally correct definition of a family within the axiomatic of Zermelo and Fraenkel. In addition, this definition fits very well to our intuitive understanding of a family: For two families $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ we have

$$(A_i)_{i \in I} = (B_i)_{i \in I} \text{ if and only if } A_i = B_i \text{ for all } i \in I \text{ (Theorem 2.6).}$$

Unfortunately, this definition of a family has also a big disadvantage: The family $(A_i)_{i \in I}$ can only be defined when we have a set $\mathcal{A}$ such that

$$A_i \in \mathcal{A} \text{ for all } i \in I.$$

This is a strong restriction: Let us consider the following example: Let $(A_i)_{i \in I}$ be a family of sets, and let C be a further set. Suppose that we want to construct the family $(B_i)_{i \in I}$ by

$$B_i := A_i \cup C \text{ for all } i \in I.$$

The expression $B_i := A_i \cup C$ looks very innocent, but a family $(B_i)_{i \in I}$ is a function $g : I \to \mathcal{B}$ into a set $\mathcal{B}$ such that

$$B_i = g(i) \text{ for all } i \in I.$$

In other words, we need the set $\mathcal{B} := \{B_i \mid i \in I\}$ for the definition of the family $(B_i)_{i \in I}$. In general, the existence of such a set $\mathcal{B}$ will not follow from the axioms of Zermelo and Fraenkel introduced so far (for more details see Remark 2.7).

The solution of this problem is to introduce a further axiom, namely the axiom of substitution (Axiom 2.10) which guarantees the existence of the set $\mathcal{B}$. For more details see Section 2.

Unions and Intersections of Families (see Section 3):

Let $(A_i)_{i\in I}$ be a family of sets, and let $\mathcal{A} := \{A_i \mid i \in I\}$. We set

$$\bigcup_{i\in I} A_i := \bigcup_{A\in\mathcal{A}} A \text{ and } \bigcap_{i\in I} A_i := \bigcap_{A\in\mathcal{A}} A.$$

There are various rules to combine unions, intersections and direct products of families of sets: We have

$$\begin{aligned}\Big(\bigcup_{i\in I} A_i\Big) \cap \Big(\bigcup_{j\in J} B_j\Big) &= \bigcup_{(i,j)\in I\times J} (A_i \cap B_j) \text{ and}\\ \Big(\bigcap_{i\in I} A_i\Big) \cup \Big(\bigcap_{j\in J} B_j\Big) &= \bigcap_{(i,j)\in I\times J} (A_i \cup B_j).\end{aligned}$$

See Proposition 3.8.

We have

$$\begin{aligned}\Big(\bigcup_{i\in I} A_i\Big) \times \Big(\bigcup_{j\in J} B_j\Big) &= \bigcup_{(i,j)\in I\times J} (A_i \times B_j) \text{ and}\\ \Big(\bigcap_{i\in I} A_i\Big) \times \Big(\bigcap_{j\in J} B_j\Big) &= \bigcap_{(i,j)\in I\times J} (A_i \times B_j).\end{aligned}$$

See Proposition 3.9.

If A_i is a subset of a set A for all elements i of an index set I and if A_i^c denotes the complement $A \setminus A_i$ of the set A_i in the set A, then we have

$$\Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c \text{ and } \Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c.$$

See Proposition 3.10.

The next question that we want to answer is the following: Suppose that we have two families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ and functions

$$f_i : A_i \to B_i \text{ for all } i \in I.$$

Does there exist a function

$$f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$$

such that

$$f(x) = f_i(x) \text{ for all } x \in A_i \text{ and all } i \in I?$$

Obviously, a necessary condition is that

$$f_i(x) = f_j(x) \text{ for all } x \in A_i \cap A_j \text{ and all } i, j \in I.$$

As we will see in Proposition 3.13, this condition is also sufficient.

The Direct Product of arbitrary many Sets (see Section 4):

The definition of families offers the possibility to define the direct product not only for two sets (as in Unit UNIONS [Garden 2020b]), but for arbitrary many sets: The intuitive definition is as follows:

Let $\left(A_i\right)_{i\in I}$ be a family of sets. Set

$$\prod_{i\in I} A_i := \left\{(z_i)_{i\in I} \mid z_i \in A_i \text{ for all } i \in I\right\}.$$

The expression $(z_i)_{i\in I}$ is a family, that is, a function $z : I \to \mathcal{A}$ from the index set I into the set $\mathcal{A} := \bigcup_{i\in I} A_i$ such that

$$z_i = z(i) \in A_i \text{ for all } i \in I.$$

The direct product $\prod_{i\in I} A_i$ therefore becomes

$$\prod_{i\in I} A_i = \{z : I \to \mathcal{A} \mid z_i := z(i) \in A_i \text{ for all } i \in I\}.$$

This is the content of Definition 4.5.

The Axiom of Choice (see Section 5):

The axiom of choice (Axiom 5.1) looks quite innocent: Given a family $\left(A_i\right)_{i\in I}$ of non-empty sets we want to be able to choose one element a_i of each set A_i. This requirement sounds more or less trivial, - one wonders whether we really need a proper axiom for it. Well, we do need the axiom of choice, but we will discuss the necessity of this axiom only in the unit Well Ordered Sets [Garden 2020f] when we will use it to prove a rather surprising result.

It remains to answer the question how the process of choosing an element a_i of each set A_i is formally defined: The axiom of choice reads as follows:

Let $\left(A_i\right)_{i\in I}$ be a family of non-empty sets. Then the product

$$\prod_{i\in I} A_i$$

is also non-empty.

An element $(a_i)_{i\in I}$ of the product $\prod_{i\in I} A_i$ fulfills the requirement

$$a_i \in A_i \text{ for all } i \in I.$$

By definition, the element $(a_i)_{i\in I}$ is a function $f : I \to \bigcup_{i\in I} A_i$ such that

$$f(i) = a_i \in A_i \text{ for all } i \in I.$$

This function $f : I \to \bigcup_{i\in I} A_i$ is called a **choice function** (see Theorem 5.3 and Remark 5.4).

Projections (see Section 6):

Let $\left(A_i\right)_{i\in I}$ be a family of non-empty sets, and let j be an element of the index set I. The **projection**

$$pr_j : \prod_{i\in I} A_i \to A_j,\ (a_i)_{i\in I} \mapsto a_j$$

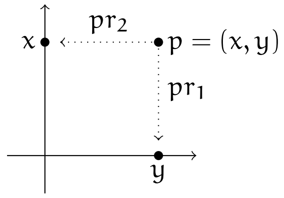
chooses the j^{th} component of the element $(a_i)_{i\in I}$ (see Definition 6.1).

The geometric meaning of a projections is as follows:

In the affine plane

$$A = \{(x, y) \mid x, y \in \mathbb{R}\}$$

the projections



$$\begin{aligned} pr_1 &: A \to \mathbb{R},\ (x, y) \mapsto x \text{ and} \\ pr_2 &: A \to \mathbb{R},\ (x, y) \mapsto y \end{aligned}$$

choose the first (resp. the second) coordinate of the point $p = (x, y)$.

Finally, given two families $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ we want to extend a family of functions

$$f_i : A_i \to B_i \text{ for all } i \in I$$

to a function

$$f : \prod_{i \in I} A_i \to \prod_{i \in I} B_i.$$

This can be done in an obvious way by setting

$$f : (x_i)_{i \in I} \mapsto \big(f(x_i)\big)_{i \in I}.$$

For more details see Proposition 6.4.

2 Families and the Axiom of Substitution

Functions:

Throughout this unit functions will play a crucial role. We therefore recall the definition of a function. For more details see Unit Functions [Garden 2020d].

2.1 Definition. Let A and B be two sets.

(a) A **function** $f : A \to B$ **from the set** A **into the set** B is a triple (f, A, B) where the set f is a subset of the direct product $A \times B$ with the following property:

For each element x of the set A, there is exactly one element y of the set B such that the pair (x, y) is contained in the set f.

(b) Let $f : A \to B$ be a function from the set A into the set B, and let x be an element of the set A. The unique element y of the set B such that the pair (x, y) is contained in the set f is denoted by $y = f(x)$. We also write $f : x \mapsto y$ or, equivalently, $f : x \mapsto f(x)$.

Families:

2.2 Definition. An **index set** I is a non-empty set.

French / German. Index set = Ensemble d'indices = Indexmenge.

2.3 Definition. Let I be an index set, and let A be a non-empty set.

(a) A function $f : I \longrightarrow A$ is called a **family of elements of the set** A.

(b) Let $f : I \longrightarrow A$ be a family of elements of the set A. We set

$$a_i := f(i) \in A \text{ for all } i \in I.$$

Instead of speaking of the family $f : I \longrightarrow A$, we usually speak of the family $(a_i)_{i \in I}$ of elements of the set A. If we want to emphasize that the objects a_i are sets, we often start with a set $\mathcal{A}$ of sets, then we consider a function $f : I \longrightarrow \mathcal{A}$, and we define $A_i := f(i)$.

French / German. Family = Famille = Familie.

2.4 Remark. Note that a family $(a_i)_{i \in I}$ is only defined if there exists a set A such that the element a_i is contained in the set A for all elements i of the set I. Otherwise, the function $f : I \longrightarrow A$ is not defined. See also Axiom 2.10 and Theorem 2.12.

Elementary Properties of Families:

For the proof of Theorem 2.6, we will need the following elementary property of functions:

2.5 Proposition. *Let A and B be two sets, and let $f : A \longrightarrow B$ and $g : A \longrightarrow B$ be two functions from the set A into the set B.*

Then we have

$$f = g \textit{ if and only if } f(x) = g(x) \textit{ for all } x \in A.$$

Proof. See Unit Functions [Garden 2020d]. □

2.6 Theorem. *Let I be an index set.*

Let $(x_i)_{i \in I}$ and $(y_i)_{i \in I}$ be two families of elements of a non-empty set A. Then we have

$$(x_i)_{i \in I} = (y_i)_{i \in I} \textit{ if and only if } x_i = y_i \textit{ for all } i \in I.$$

Proof. By definition of a family, there exist two functions $f : I \longrightarrow A$ and $g : I \longrightarrow A$ such that

$$x_i = f(i) \text{ and } y_i = g(i) \text{ for all } i \in I.$$

By Proposition 2.5, we have

$$(x_i)_{i \in I} = (y_i)_{i \in I} \Leftrightarrow f = g \Leftrightarrow f(i) = g(i) \text{ for all } i \in I \Leftrightarrow x_i = y_i \text{ for all } i \in I.$$

□

The Axiom of Substitution:

2.7 Remark. Let $\mathcal{A}$ and B be two sets. We want to express something like *For each set A of the set $\mathcal{A}$, let* $B_A := A \cup B$.

Unfortunately, this is in general not possible since the definition of the family $(B_A)_{A \in \mathcal{A}}$ requires a function

$$f : \mathcal{A} \longrightarrow \{B_A \mid A \in \mathcal{A}\}$$

from the set $\mathcal{A}$ into the set $\mathcal{B} := \{B_A \mid A \in \mathcal{A}\}$, but the existence of the set $\mathcal{B}$ is not guaranteed. The axiom of substitution solves this problem.

The axiom of substitution is one of the axioms of Zermelo and Fraenkel. The complete list of the axioms of Zermelo and Fraenkel is as follows:

ZFC-0:	Basic Axiom
ZFC-1:	Axiom of Extension
ZFC-2:	Axiom of Existence
ZFC-3:	Axiom of Specification
ZFC-4:	Axiom of Pairing
ZFC-5:	Axiom of Unions
ZFC-6:	Axiom of Powers
ZFC-7:	Axiom of Foundation
ZFC-8:	Axiom of Substitution
ZFC-9:	Axiom of Choice
ZFC-10:	Axiom of Infinity

These axioms are explained in the units UNIVERSE [Garden 2020a] (ZFC-0 to ZFC-3), UNIONS [Garden 2020b] (ZFC-4 to ZFC-7) and SUCCESSOR SETS [Garden 2020e] (ZFC-10). The axiom of substitution (ZFC-8) and the axiom of choice (ZFC-9) are explained in the present unit (see Axiom 2.10 and Axiom 5.1).

2.8 Definition. A sentence $\varphi(x, y)$ in the two variables x and y is called **functional** if for each element x there exists exactly one element y.

French / German. Functional Sentence = Terme Fonctionel = Funktionaler Ausdruck.

2.9 Example. Let A be a set. Then the sentence $\varphi(X, Y) := [Y = X \cup A]$ is an example of a functional sentence.

2.10 Axiom. (ZFC-8: Axiom of Substitution or Axiom of Replacement) For each functional sentence $\varphi = \varphi(x, y)$ and for each set A there exists a set B such that

$$y \in B \text{ if and only if } \exists\, x \in A : \varphi(x, y).$$

French / German. Axiom of substitution (or axiom of replacement) = Axiome de substitution (or axiome de remplacement) = Ersetzungsaxiom.

2.11 Example. Let C be a set, and let $\varphi = \varphi(A, B)$ be the sentence

$$\varphi(A, B) := [B = A \cup C].$$

Let $\mathcal{A}$ be a set of sets. For each set A of the set $\mathcal{A}$, the set $B := A \cup C$ exists (axiom of unions, see Unit UNIONS [Garden 2020b]). It follows that the sentence $\varphi(A, B) := [B = A \cup C]$ is functional.

By the axiom of substitution, there exists a set $\mathcal{B}$ such that

$$B \in \mathcal{B} \quad \Leftrightarrow \quad \exists\, A \in \mathcal{A} \text{ such that } \varphi(A, B)$$
$$\Leftrightarrow \quad \exists\, A \in \mathcal{A} \text{ such that } B = A \cup C.$$

In other words, the set

$$\mathcal{B} := \{A \cup C \mid A \in \mathcal{A}\}$$

exists. We are now able to define the family $\left(B_A\right)_{A \in \mathcal{A}}$ via the function $f : \mathcal{A} \to \mathcal{B}$, $f : A \mapsto A \cup C$, that is, the family $(B_A)_{A \in \mathcal{A}}$ where

$$B_A := f(A) = A \cup C \text{ for all } A \in \mathcal{A}.$$

2.12 Theorem. *Let* I *be an index set, let* $\varphi = \varphi(i)$ *be a sentence, and suppose that the sentence*

$$\psi(i, A) := [A = \varphi(i)]]$$

is functional. Then the family $(A_i)_{i \in I}$ *with* $A_i := \varphi(i)$ *exists.*

Proof. By assumption, the sentence

$$\psi(A, i) := [A = \varphi(i)]$$

is functional. It follows from the axiom of substitution (Axiom 2.10) that there exists a set $\mathcal{A}$ such that

$$A \in \mathcal{A} \text{ if and only if } \exists\, i \in I \text{ such that } A = \varphi(i).$$

Let the function $f : I \to \mathcal{A}$ be defined as follows: For each element i of the set I, let $f(i)$ be the set A of the set $\mathcal{A}$ such that $A = \varphi(i)$. Set $A_i := f(i)$. Then the family $\left(A_i\right)_{i \in I}$ has the required properties. □

2.13 Example. Let $\mathcal{A}$ be a set of sets. For each set A of the set $\mathcal{A}$, let $\varphi(A) := A \times A$. Since the sentences $\varphi = \varphi(A)$ fulfills the assumptions of Theorem 2.12, we may define

$$C_A := \varphi(A) = A \times A \text{ for all } A \in \mathcal{A}.$$

Historical Notes:

The axiomatic of mathematics based on set theory is mainly due to Ernst Zermelo based on earlier work of Richard Dedekind and Georg Cantor. He published his first version of his axiomatic in 1908 [Zermelo 1908]. In this article the axiom of substitution is still missing as has been noticed by Abraham Fraenkel[1].

I. Die überaus scharfsinnigen Untersuchungen Zermelos sollen hierdurch nicht umgestoßen, sondern nur vervollständigt und befestigt werden [...]

Die sieben Zermeloschen Axiome reichen nicht aus zur Begründung der Mengenlehre.

Zum Nachweis dieser Behauptung diene etwa das folgende einfache Beispiel: Es sei Z_0 *die in [Zermelo 1908] definierte und als existierend nachgewiesene Menge (Zahlenreihe).*

[1] Fraenkel changed his given name to Abraham.

Die Potenzmenge $\mathcal{P}(Z_0)$ *(Menge aller Untermengen von* Z_0*) werde mit* Z_1*,*$\mathcal{P}(Z_1)$ *mit* Z_2 *bezeichnet usw. Dann gestatten die Axiome [...] nicht die Bildung der Menge* $\{Z_0, Z_1, \ldots\}$*, [...] Diese bisher nicht bemerkte Lücke der Zermeloschen Begründung ist durch Hinzufügung eines neuen Axioms oder Erweiterung eines vorhandenen auszufüllen [...]*

Ersetzungsaxiom. *Ist* M *eine Menge und wird jedes Element von* M *durch "ein Ding des Bereiches* $\mathcal{B}$*" (vgl. Z. S. 262) ersetzt, so geht* M *wiederum in eine Menge über.*

See [Fraenkel 1922, pp. 230 - 231].

The extremely ingenious investigations of Zermelo are not meant to be overturned, but only to be completed and fortified. [...]

The seven axioms of Zermelo are not sufficient to justify set theory.

The following simple example serves to prove this claim: Let Z_0 *be the set defined in [Zermelo 1908] and whose existence has been shown (series of numbers). Let us denote the power set* $\mathcal{P}(Z_0)$ *(set of all subsets of* Z_0*) by* Z_1 *the set* $\mathcal{P}(Z_1)$ *by* Z_2 *etc. Then the axioms [...] do not allow the formation of the set* $\{Z_0, Z_1, \ldots\}$*, [...] This hitherto undetected gap in Zermelo's reasoning has to be filled in by adding a new axiom or expanding an existing one.*

Axiom of Substitution. *If* M *is a set and every element of* M *is replaced by "a thing of the domain* $\mathcal{B}$*" (see Z. p. 262), then* M *turns into a set.*

(Translation by the author.)

The domain $\mathcal{B}$ is the mathematical universe (see Unit UNIVERSE [Garden 2020a]).

Zermelo published a second version of his axiomatic which is today called the axiomatic ZFC of Zermelo and Fraenkel (the letter C stands for the axiom of choice) in 1930 [Zermelo 1930]. He included the axiom of substitution in this article:

E) **Axiom der Ersetzung:** *Ersetzt man die Elemente* x *einer Menge* m *eindeutig durch beliebige Elemente* x' *des Bereiches, so enthält dieser auch eine Menge* m'*, welche alle diese* x' *zu Elementen hat.*

See [Zermelo 1930, pp. 30-31].

E) **Axiom of Substitution:** *If the elements* x *of a set* m *are uniquely replaced by any elements* x *of the domain, the domain also contains a set* m'*, which has all these* x' *as elements.*

(Translation by the author.)

3 Unions and Intersections of Families

Unions and Intersections:

3.1 Definition. Let I be an index set, let $(A_i)_{i \in I}$ be a family of sets, and let $\mathcal{A} := \{A_i \mid i \in I\}$. We set

$$\bigcup_{i \in I} A_i := \bigcup_{A \in \mathcal{A}} A \text{ and } \bigcap_{i \in I} A_i := \bigcap_{A \in \mathcal{A}} A.$$

3.2 Remark. Note that, by Definition 2.2, an index set is always non-empty. It follows that the set $\mathcal{A} := \{A_i \mid i \in I\}$ is also non-empty. As a consequence the set $\bigcap_{A\in\mathcal{A}} A$ is defined.

3.3 Proposition. *Let* I *be an index set, and let* $(A_i)_{i\in I}$ *be a family of sets.*

(a) We have

$$x \in \bigcup_{i\in I} A_i \text{ if and only if } x \in A_j \text{ for at least one element } j \in I.$$

(b) We have

$$x \in \bigcap_{i\in I} A_i \text{ if and only if } x \in A_i \text{ for all } i \in I.$$

Proof. Let $\mathcal{A} := \{A_i \mid i \in I\}$. By Definition 3.1, we have

$$\bigcup_{i\in I} A_i = \bigcup_{A\in\mathcal{A}} A \text{ and } \bigcap_{i\in I} A_i = \bigcap_{A\in\mathcal{A}} A.$$

(a) The assertion follows from the fact that

$$x \in \bigcup_{A\in\mathcal{A}} A \text{ if and only if } x \in A \text{ for at least one set } A \in \mathcal{A}.$$

For more details see Unit UNIONS [Garden 2020b].

(b) The assertion follows from the fact that

$$x \in \bigcap_{A\in\mathcal{A}} A \text{ if and only if } x \in A \text{ for all } A \in \mathcal{A}.$$

For more details see Unit UNIONS [Garden 2020b]. □

3.4 Proposition. *Let* I *be an index set, and let* $(A_i)_{i\in I}$ *be a family of sets.*

(a) Let J *be a subset of the set* I*. Then we have*

$$\bigcup_{j\in J} A_j \subseteq \bigcup_{i\in I} A_i.$$

(b) Let J *be a non-empty subset of the set* I*. Then we have*

$$\bigcap_{j\in J} A_j \supseteq \bigcap_{i\in I} A_i.$$

(c) Let A *and* B *be two sets such that* $A \subseteq A_i \subseteq B$ *for all elements* i *of the set* I*. Then we have*

$$A \subseteq \bigcup_{i\in I} A_i \subseteq B \text{ and } A \subseteq \bigcap_{i\in I} A_i \subseteq B.$$

(d) Let $(B_i)_{i\in I}$ *be a family of sets such that* $A_i \subseteq B_i$ *for all elements* i *of the set* I*. Then we have*

$$\bigcup_{i\in I} A_i \subseteq \bigcup_{i\in I} B_i \text{ and } \bigcap_{i\in I} A_i \subseteq \bigcap_{i\in I} B_i.$$

Proof. The proof follows from Proposition 3.3. □

3.5 Proposition. *Let I and J be two index sets, and let $I = \bigcup_{j\in J} I_j$ for a family $(I_j)_{j\in J}$ of non-empty subsets of the set I. Then we have*

$$\bigcup_{i\in I} A_i = \bigcup_{j\in J}\Big(\bigcup_{k\in I_j} A_k\Big) \text{ and } \bigcap_{i\in I} A_i = \bigcap_{j\in J}\Big(\bigcap_{k\in I_j} A_k\Big).$$

Proof. (a) Let x be an element of the set $\bigcup_{i\in I} A_i$. Then there exists an element i of the set I such that the element x is contained in the set A_i. Since $I = \bigcup_{j\in J} I_j$, there exists an element j of the set J such that the set I_j contains the element i. It follows that

$$x \in A_i \subseteq \bigcup_{k\in I_j} A_k.$$

It follows that

$$\bigcup_{i\in I} A_i \subseteq \bigcup_{j\in J}\Big(\bigcup_{k\in I_j} A_k\Big).$$

The inclusion $\bigcup_{j\in J}\big(\bigcup_{k\in I_j} A_k\big) \subseteq \bigcup_{i\in I} A_i$ is obvious.

(b) follows as (a). □

3.6 Remark. Proposition 3.5 generalizes the associative laws

$$(A \cup B) \cup C = A \cup (B \cup C) \text{ and } (A \cap B) \cap C = A \cap (B \cap C)$$

introduced in Unit Unions [Garden 2020b].

3.7 Proposition. *Let I be an index set, and let $(A_i)_{i\in I}$ be a family of sets.*

(a) We have

$$A \cap \Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I}(A \cap A_i).$$

(b) We have

$$A \cup \Big(\bigcap_{i\in I} A_i\Big) = \bigcap_{i\in I}(A \cup A_i).$$

Proof. (a) Let x be an element of the set $A \cap \big(\bigcup_{i\in I} A_i\big)$. Then the element x is contained in the set A, and there exists an element i of the set I such that the element x is contained in the set A_i. It follows that the element x is contained in the set $\bigcup_{i\in I}(A \cap A_i)$.

Conversely, let x be an element of the set $\bigcup_{i\in I}(A \cap A_i)$. Then there exists an element i of the set I such that the element x is contained in the set $A \cap A_i$, that is, in the set A and in the set A_i. It follows that the element x is contained in the set $A \cap \big(\bigcup_{i\in I} A_i\big)$.

(b) Let x be an element of the set $A \cup \big(\bigcap_{i\in I} A_i\big)$. If the element x is contained in the set A, then the element x is obviously contained in the set $\bigcap_{i\in I}(A \cup A_i)$. If the element x is contained in the set $\big(\bigcap_{i\in I} A_i\big)$, then the element x is obviously contained in the set $\bigcap_{i\in I}(A \cup A_i)$. It follows that the element x is contained in the set $\big(\bigcap_{i\in I} A_i\big)$.

Conversely, let x be an element of the set $\bigcap_{i\in I}(A\cup A_i)$. Suppose that the element x is not contained in the set $\left(\bigcap_{i\in I} A_i\right)$. Then there exists an element k of the set I such that the element x is not contained in the set A_k. Since

$$x\in\bigcap_{i\in I}(A\cup A_i)\subseteq A\cup A_k,$$

it follows that the element x is contained in the set $A\cap A_k$. Since the element x is not contained in the set A_k, the element x is contained in the set A. Altogether, the element x is contained in the set $A\cup\left(\bigcap_{i\in I} A_i\right)$. □

3.8 Proposition. *Let I and J be two index sets, and let $(A_i)_{i\in I}$ and $(B_j)_{j\in J}$ be two families of sets.*

(a) We have

$$\Big(\bigcup_{i\in I} A_i\Big)\cap\Big(\bigcup_{j\in J} B_j\Big)=\bigcup_{(i,j)\in I\times J}(A_i\cap B_j).$$

(b) We have

$$\Big(\bigcap_{i\in I} A_i\Big)\cup\Big(\bigcap_{j\in J} B_j\Big)=\bigcap_{(i,j)\in I\times J}(A_i\cup B_j).$$

Proof. (a) Let x be an element of the set $\left(\bigcup_{i\in I} A_i\right)\cap\left(\bigcup_{j\in J} B_j\right)$. Then there exists an element i of the set I and an element j of the set J such that the element x is contained in the set A_i and B_j. It follows that the element x is contained in the set $\bigcup_{(i,j)\in I\times J}(A_i\cap B_j)$.

Conversely, let x be an element of the set $\bigcup_{(i,j)\in I\times J}(A_i\cap B_j)$. Then there exists an element i of the set I and an element j of the set J such that the element x is contained in the set $A_i\cap B_j$, that is, in the set A_i and B_j. It follows that the element x is contained in the set $\left(\bigcup_{i\in I} A_i\right)\cap\left(\bigcup_{j\in J} B_j\right)$.

(b) Let x be an element of the set $\left(\bigcap_{i\in I} A_i\right)\cup\left(\bigcap_{j\in J} B_j\right)$. If the element x is contained in the set $\left(\bigcap_{i\in I} A_i\right)$, then the element x is obviously contained in the set $\bigcap_{(i,j)\in I\times J}(A_i\cup B_j)$. If the element x is contained in the set $\left(\bigcap_{j\in J} B_j\right)$, then the element x is obviously contained in the set $\bigcap_{(i,j)\in I\times J}(A_i\cup B_j)$. It follows that the element x is contained in the set $\bigcap_{(i,j)\in I\times J}(A_i\cup B_j)$.

Conversely, let x be an element of the set $\bigcap_{(i,j)\in I\times J}(A_i\cup B_j)$. Suppose that the element x is not contained in the set $\left(\bigcap_{i\in I} A_i\right)$. Then there exists an element k of the set I such that the element x is not contained in the set A_k. Since

$$\bigcap_{(i,j)\in I\times J}(A_i\cup B_j)\subseteq\bigcap_{(i,j)\in\{k\}\times J}(A_i\cup B_j)=\bigcap_{j\in J}(A_k\cup B_j),$$

it follows that the element x is contained in the set

$$\bigcap_{j\in J}(A_k\cup B_j)=A_k\cup\Big(\bigcap_{j\in J} B_j\Big)\ \text{(Proposition 3.7)}.$$

Since the element x is not contained in the set A_k, the element x is contained in the set $\left(\bigcap_{j\in J} B_j\right)$. □

Unions, Intersections and Direct Products:

3.9 Proposition. *Let* I *and* J *be two index sets, and let* $(A_i)_{i\in I}$ *and* $(B_j)_{j\in J}$ *be two families of sets.*

(a) We have

$$\Big(\bigcup_{i\in I} A_i\Big) \times \Big(\bigcup_{j\in J} B_j\Big) = \bigcup_{(i,j)\in I\times J} (A_i \times B_j).$$

(b) We have

$$\Big(\bigcap_{i\in I} A_i\Big) \times \Big(\bigcap_{j\in J} B_j\Big) = \bigcap_{(i,j)\in I\times J} (A_i \times B_j).$$

Proof. The proof follows from Proposition 3.3. □

Unions, Intersections and Complements:

3.10 Proposition. *Let* I *be an index set, let* A *be a set, and let* $(A_i)_{i\in I}$ *be a family of subsets of the set* A*. For a subset* X *of the set* A *denote by* $X^c := A \setminus X$ *the complement of the set* X *in the set* A*.*

(a) We have

$$\Big(\bigcup_{i\in I} A_i\Big)^c = \bigcap_{i\in I} A_i^c.$$

(b) We have

$$\Big(\bigcap_{i\in I} A_i\Big)^c = \bigcup_{i\in I} A_i^c.$$

Proof. (a) Let x be an element of the set $\big(\bigcup_{i\in I} A_i\big)^c$. Then the element x is not contained in any set A_i implying that the element x is contained in the set $\bigcap_{i\in I} A_i^c$.

Conversely, let x be an element of the set $\bigcap_{i\in I} A_i^c$. Assume that the element x is not contained in the set $\big(\bigcup_{i\in I} A_i\big)^c$. Then the element x is contained in the set $\bigcup_{i\in I} A_i$. Hence, there exists an element j of the set I such that the element x is contained in the set A_j, in contradiction to the assumption that the element x is contained in the set $\bigcap_{i\in I} A_i^c$.

(b) It follow from (a) that

$$\Big(\bigcup_{i\in I} A_i^c\Big)^c = \bigcap_{i\in I} (A_i^c)^c = \bigcap_{i\in I} A_i, \text{ hence } \bigcup_{i\in I} A_i^c = \Big(\bigcap_{i\in I} A_i\Big)^c.$$

□

3.11 Remark. Note that Proposition 3.10 generalizes de Morgan's laws explained in Unit Unions [Garden 2020b].

Unions, Intersections and Inverse Images:

3.12 Proposition. *Let* I *be an index set, let* A *and* B *be two sets, and let* $(B_i)_{i\in I}$ *a family of subsets of the set* B*. Let* $f : A \longrightarrow B$ *be a function from the set* A *into a set* B*.*

Then we have

$$f^{-1}\Big(\bigcup_{i\in I} B_i\Big) = \bigcup_{i\in I} f^{-1}(B_i) \text{ and } f^{-1}\Big(\bigcap_{i\in I} B_i\Big) = \bigcap_{i\in I} f^{-1}(B_i).$$

Proof. Let x be an element of the set $f^{-1}\big(\bigcup_{i\in I} B_i\big)$. Then the element $f(x)$ is contained in the set $\bigcup_{i\in I} B_i$. Hence, there exists an element j of the set I such that the element $f(x)$ is contained in the set B_j, that is, the element x is contained in the set $f^{-1}(B_j)$. In particular, we have

$$x \in \bigcup_{i\in I} f^{-1}(B_i).$$

Conversely, let x be an element of the set $\bigcup_{i\in I} f^{-1}(B_i)$. Then there exists an element j of the set I such that the element x is contained in the set $f^{-1}(B_j)$, that is, the element $f(x)$ is contained in the set B_j. In particular, the element $f(x)$ is contained in the set $\bigcup_{i\in I} B_i$ implying that

$$x \in f^{-1}\Big(\bigcup_{i\in I} B_i\Big).$$

The second part follows in the same way. □

Extensions of Functions:

3.13 Proposition. *Let I be an index set, and let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of non-empty sets. For each element i of the set I let $f_i : A_i \to B_i$ be a function from the set A_i into the set B_i.*

Suppose that for each two elements i and j of the set I, we have $f_i(x) = f_j(x)$ for all elements x of the set $A_i \cap A_j$.[a]

(a) There exists a function

$$f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$$

such that $f|_{A_i} = f_i$ for all elements i of the set I.

(b) If the functions $f_i : A_i \to B_i$ are surjective for all elements i of the set I, then the function $f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also surjective.

[a] Note that this implies that $f_i(x) = f_j(x) \in B_i \cap B_j$ for all $x \in A_i \cap A_j$.

Proof. (a) Let i be an element of the set I. Then the function $f_i : A_i \to B_i$ is a subset of the direct product $A_i \times B_i$ with the property that for each element x_i of the set A_i, there exists exactly one element y_i of the set B_i such that the pair (x_i, y_i) is contained in the set f_i. Set

$$f := \bigcup_{i\in I} f_i.$$

Then the set f is a subset of the set

$$\bigcup_{i\in I}(A_i \times B_i) \subseteq \bigcup_{(i,j)\in I\times J} (A_i \times B_j) = \Big(\bigcup_{i\in I} A_i\Big) \times \Big(\bigcup_{i\in I} B_i\Big) \text{ (Proposition 3.9).}$$

Let x be an element of the set $\bigcup_{i\in I} A_i$. Since there exists an element i of the set I such that the element x is contained in the set A_i, we have

$$\big(x, f_i(x)\big) \in f_i \subseteq \bigcup_{i\in I} f_i = f.$$

Assume that there exist two different pairs (x, y_1) and (x, y_2) in the set f. Since each of the sets f_i are functions, there must exist two indices i and j such that

$$(x, y_1) \in f_i \text{ and } (x, y_2) \in f_j.$$

It follows that the element x is contained in the set $A_i \cap A_j$, and we get

$$y_1 = f_i(x) = f_j(x) = y_2,$$

a contradiction.

Hence, the set f is a function, and we have $f(x_i) = f_i(x_i)$ for all elements x_i of the set A_i.

(b) Let y be an element of the set $\bigcup_{i\in I} B_i$. Then there exists an element j of the set I such that the element y is contained in the set B_j. Since the function $f_j : A_j \to B_j$ is surjective, there exists an element x of the set A_j such that $f(x) = f_j(x) = y$. □

3.14 Proposition. *Let I be an index set, let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of non-empty sets, and suppose that the sets $(A_i)_{i\in I}$ and the sets $(B_i)_{i\in I}$ are pairwise disjoint. For each element i of the set I let $f_i : A_i \to B_i$ be a function from the set A_i into the set B_i.*

(a) There exists a function

$$f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$$

such that

$$f(x) = f_i(x) \text{ for all } x \in A_i \text{ and for all } i \in I.$$

(b) If the functions $f_i : A_i \to B_i$ are injective for all elements i of the set I, then the function $f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also injective.

(c) If the functions $f_i : A_i \to B_i$ are surjective for all elements i of the set I, then the function $f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also surjective.

(d) If the functions $f_i : A_i \to B_i$ are bijective for all elements i of the set I, then the function $f : \bigcup_{i\in I} A_i \to \bigcup_{i\in I} B_i$ is also bijective.

Proof. (a) follows from Proposition 3.13.

(b) Suppose that $f(x) = f(x')$ for two elements x and x' of the set $\bigcup_{i\in I} A_i$. Then there exists an element i of the set I such that the element $f(x) = f(x')$ is contained in the set B_i.

Since the sets $(B_i)_{i\in I}$ are pairwise disjoint, it follows that the elements x and x' are contained in the set A_i and that $f_i(x) = f(x) = f(x') = f_i(x')$. Since the function $f_i : A_i \to B_i$ is injective, we have $x = x'$.

(c) follows from Proposition 3.13.

(d) follows from (b) and (c). □

4 The Direct Product of Arbitrary Many Sets

An Alternative Description of the Direct Product of two Sets:

We recall the definition of the direct product of two sets explained in Union Direct Products [Garden 2020c]:

4.1 Definition. Let a and b be two sets.

The **ordered pair** (a, b) is defined by $(a, b) := \big\{\{a\}, \{a, b\}\big\}$.

4.2 Definition. Let A and B be two sets. Set

$$A \times B := \{x \in \mathcal{P}\big(\mathcal{P}(A \cup B)\big) \mid \exists\, a \in A\, \exists\, b \in B \text{ s.t. } x = (a, b)\}.$$

(a) The set $A \times B$ is called the **direct product of the sets** A **and** B or, equivalently, the **Cartesian product of the sets** A **and** B.

(b) We write $A \times B := \{(a, b) \mid a \in A \text{ and } b \in B\}$ for short.

4.3 Proposition. *Let A and B be two non-empty sets, let $I := \{A, B\}$, and let*

$$Z := \{z : I \to A \cup B \mid z_A := z(A) \in A \textit{ and } z_B := z(B) \in B\}$$

be the set of the functions $z : I = \{A, B\} \to A \cup B$ such that the elements $z(A)$ and $z(B)$ are contained in the sets A and B, respectively.

Then the mapping

$$\gamma : Z \to A \times B;\ \gamma : z \mapsto (z_A, z_B)$$

is a bijective mapping from the set Z onto the direct product $A \times B$.

Proof. Step 1. *The mapping $\gamma : Z \to A \times B$ is surjective:*

For, let (a, b) be an element of the set $A \times B$. Let $z : I \to A \cup B$ be the function defined by $z(A) := a$ and $z(B) := b$. Then the function $z : I \to A \cup B$ is an element of the set Z, and we have

$$\gamma : z \mapsto (z_A, z_B) = \big(z(A), z(B)\big) = (a, b).$$

Step 2. *The mapping $\gamma : Z \to A \times B$ is injective:*

Let z and z' be two functions of the set Z such that $\gamma(z) = \gamma(z')$. Then we have $(z_A, z_B) = (z'_A, z'_B)$ implying that

$$z(A) = z_A = z'_A = z'(A) \text{ and } z(B) = z_B = z'_B = z'(B).$$

By Proposition 2.5, it follows that $z = z'$. □

4.4 Remark. Let A and B be two non-empty sets, let $I := \{A, B\}$, and let

$$Z := \{z : I \to A \cup B \mid z_A := z(A) \in A \text{ and } z_B := z(B) \in B\}$$

be the set of the functions $z : I = \{A, B\} \to A \cup B$ such that the elements $z(A)$ and $z(B)$ are contained in the sets A and B, respectively.

Identifying the sets Z and $A \times B$ (see Proposition 4.3) we can say that the product $A \times B$ consists of the families $(z_i)_{i \in \{A,B\}}$ such that the elements $z(A)$ and $z(B)$ are contained in the sets A and B, respectively. If we denote the family $(z_i)_{i \in \{A,B\}}$ by (z_A, z_B), we get

$$A \times B = \{(z_A, z_B) \mid z_A \in A \text{ and } z_B \in B\}.$$

We will use this alternative definition of a direct product of two sets as a template for defining the direct product of arbitrary many sets.

4.5 Definition. Let I be an index set, and let $(A_i)_{i\in I}$ be a family of sets.

(a) The **direct product** A **of the sets** A_i is defined as follows:

$$\begin{aligned} A &:= \Big\{z : I \to \bigcup_{i\in I} A_i \mid z_i := z(i) \in A_i \text{ for all } i \in I\Big\} \\ &= \Big\{(z_i)_{i\in I} \mid z_i \in A_i \text{ for all } i \in I\Big\}. \end{aligned}$$

(b) The direct product A of the sets A_i is denoted by

$$A := \prod_{i\in I} A_i.$$

French / German. Direct product = Produit direct = Direktes Produkt.

4.6 Remark. The definition of the direct product has the little disadvantage that the direct product of two sets is defined in two different ways. On the one hand we have the definition

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

according to Definition 4.1. On the other hand, we have Definition 4.5. Of course, it would have been possible to call the first variant the *weak direct product* and the second variant the *direct product*, but traditionally both variants are called the direct product and are used simultaneously.

Elementary Properties of the Direct Product:

4.7 Proposition. *Let* I *be an index set, let* $(A_i)_{i\in I}$ *be a family of sets, and let* $a = \big(a_i\big)_{i\in I}$ *and* $b = \big(b_i\big)_{i\in I}$ *be two elements of the set* $A := \prod_{i\in I} A_i$. *Then we have*

$$a = b \text{ if and only if } a_i = b_i \text{ for all } i \in I.$$

Proof. The proof follows from Theorem 2.6. □

4.8 Proposition. *Let* I *be an index set, let* $(A_i)_{i\in I}$ *be a family of sets, and let* $A := \prod_{i\in I} A_i$ *be the direct product of the sets* A_i.

If $A_j = \emptyset$ *for at least one element* j *of the set* I, *then we have* $A = \emptyset$.

Proof. Assume that $A \neq \emptyset$. Then there exists a function $z : I \to \bigcup_{i\in I} A_i$ such that the elements $z(i)$ are contained in the sets A_i for all elements i of the set I. In particular, the element $z(j)$ is contained in the set A_j, in contradiction to the assumption that $A_j = \emptyset$. □

4.9 Remark. The converse (*If the sets* A_i *are non-empty for all elements* i *of the set* I, *then the direct product* A *is also non-empty*) will be introduced as a further axiom, the so-called axiom of choice (see Axiom 5.1).

5 The Axiom of Choice

The Axiom of Choice:

5.1 Axiom. (ZFC-9: The Axiom of Choice) *Let I be an index set, and let $(X_i)_{i \in I}$ be a family of non-empty sets. Then the direct product*

$$X := \prod_{i \in I} X_i$$

is also a non-empty set.

French / German. Axiom of choice = Axiome du choix = Auswahlaxiom.

Elementary Conclusions from the Axiom of Choice:

5.2 Proposition. *Let I be an index set, and let $(X_i)_{i \in I}$ be a family of non-empty sets. Then there exists a family $(x_i)_{i \in I}$ of elements such that the element x_i is contained in the set X_i for all elements i of the set I.*

Proof. Let

$$X := \prod_{i \in I} X_i = \{(x_i)_{i \in I} \mid x_i \in X_i\}.$$

By the axiom of choice, we have $X \neq \emptyset$. The assertion follows. □

5.3 Theorem. *Let $\mathcal{C}$ be a non-empty set of non-empty sets. Then there exists a function*

$$f : \mathcal{C} \to \bigcup_{C \in \mathcal{C}} C$$

such that the element $f(X)$ is contained in the set X for all elements X of the set $\mathcal{C}$.

Proof. It follows from the axiom of choice that the direct product $A := \prod_{X \in \mathcal{C}} X$ is non-empty. Let f be an element of the set A. By definition of the direct product (Definition 4.5, see also Definition 2.3), the element f is a function

$$f : \mathcal{C} \to \bigcup_{X \in \mathcal{C}} X$$

such that $f(X) \in X$ for all elements X of the set $\mathcal{C}$. □

5.4 Remark. Theorem 5.3 motivates the name of the axiom of choice. The function

$$f : \mathcal{C} \to \bigcup_{X \in \mathcal{C}} X$$

chooses from each (non-empty) set X of the (non-empty) set $\mathcal{C}$ an element $f(X)$. The function f is sometimes called a **choice function**.

5.5 Theorem. *Let S be a non-empty set. Then there exists a function*

$$f : \mathcal{P}(S) \setminus \{\emptyset\} \to S$$

from the set of the non-empty subsets of the set S into the set S such that the element $f(X)$ is contained in the set X for all non-empty subsets X of the set S.

Proof. The assertion follows from Theorem 5.3 by setting

$$\mathcal{C} := \{X \in \mathcal{P}(S) \mid X \neq \emptyset\}.$$

□

Historical Notes:

Ernst Zermelo used the axiom of choice in his proof of the theorem that every set can be endowed with a well ordering (for more details about well ordered sets see Unit Well Ordered Sets). He explicitly points out that the proof of his theorem relies on the axiom of choice:

Der vorliegende Beweis beruht auf der Voraussetzung, dass [...] es auch für eine unendliche Gesamtheit von Mengen immer Zuordnungen gibt, bei denen jeder Menge eines ihrer Elemente entspricht, oder formal ausgedrückt, dass das Produkt einer unendlichen Gesamtheit von Mengen, deren jede mindestens ein Element enthält, selbst von 0 verschieden ist.

[...]

Dieses logische Prinzip lässt sich zwar nicht auf ein noch einfacheres zurückführen, wird aber in der mathematischen Deduktion überall unbedenklich angewendet.

See [Zermelo 1904, p. 516].

The present proof rests upon the assumption that [...] even for an infinite totality of sets there are always mappings that associate with every set one of its elements, or, expressed formally, that the product of an infinite totality of sets, each containing at least one element itself differs from zero.

[...]

This logical principle cannot, to be sure, be reduced to a still simpler one, but it is applied without hesitation everywhere in mathematical deductions.

See [Zermelo 1967b, p. 141].

"Differs from 0" just means that the set is not empty. In 1908 Zermelo published his first list of axioms which contains the axiom of choice:

Um nun den Satz zu gewinnen, dass ein Produkt mehrerer Mengen nur dann verschwinden (d.h. der Nullmenge gleich sein) kann, wenn ein Faktor verschwindet, brauchen wir ein weiteres Axiom.

Axiom VI. *Ist T eine Menge, deren sämtliche Elemente von 0 verschiedene Mengen und untereinander elementfremd sind, so enthält ihre Vereinigung $\mathfrak{S}T$ mindestens eine Untermenge S_1, welche mit jedem Elemente von T ein und nur ein Element gemein hat. (Axiom der Auswahl)*

Man kann das Axiom auch so ausdrücken, dass man sagt, es sei immer möglich, aus jedem Elemente $M, N, R, \dots$ *von* T *ein einzelnes Element* $m, n, r, \dots$ *auszuwählen und alle diese Elemente zu einer Menge zu vereinigen.*

See [Zermelo 1908, p. 266].

In order, now, to obtain the theorem that the product of several sets can vanish (that is, be equal to the null set) only if a factor vanishes we need a further axiom.

Axiom VI. (Axiom of Choice) If T is a set whose elements all are sets that are different from 0 and mutually disjoint, its union $\mathfrak{S}T$ includes at least one subset S_1 having one and only one element in common with each element of T.

We can also express this axiom by saying that ist is always possible to choose a single element from each element $M, N, R, \dots$ of T and to combine all the chosen elements $m, n, r, \dots$ into a set S_1.

See [Zermelo 1967a, p. 204].

6 Projections

Definition of Projections:

6.1 Definition. Let I be an index set, let $(A_i)_{i \in I}$ be a family of sets, and let $A := \prod_{i \in I} A_i$ be the direct product of the sets A_i.

(a) For an element j of the set I, we define the function

$$\mathrm{pr}_j : A \longrightarrow A_j \text{ by } \mathrm{pr}_j : x = (x_i)_{i \in I} \longmapsto x_j.$$

The function $\mathrm{pr}_j : A \longrightarrow A_j$ is called the **projection from the set A onto the set A_j**.

(b) For a subset J of the set I, we define the function

$$\mathrm{pr}_J : A \longrightarrow \prod_{j \in J} A_j \text{ by } \mathrm{pr}_J : x = (x_i)_{i \in I} \longmapsto (x_j)_{j \in J}.$$

The function $\mathrm{pr}_J : A \longrightarrow A_j$ is called the **projection from the set A onto the set** $\prod_{j \in J} A_j$.

French / German. Projection = Projection = Projektion.

Elementary Properties of Projections:

6.2 Proposition. *Let I be an index set, let $(A_i)_{i \in I}$ be a family of non-empty sets, and let $A := \prod_{i \in I} A_i$ be the direct product of the sets A_i.*

(a) Let j be an element of the set I. Then the projection $\mathrm{pr}_j : A \longrightarrow A_j$ is surjective.

(b) Let J be a subset of the set I. Then the projection $\mathrm{pr}_J : A \longrightarrow \prod_{j \in J} A_j$ is surjective.

Proof. (a) Let y_j be an element of the set A_j. By the axiom of choice (Axiom 5.1), there exists an element $(x_i)_{i \in I}$ of the direct product A. Note that the element $(x_i)_{i \in I}$ is a function

$f : I \to \bigcup_{i \in I} A_i$ such that the element $x_i := f(i)$ is contained in the set A_i for all elements i of the set I (Definition 4.5).

Define the function $g : I \to \bigcup_{i \in I} A_i$ by

$$g(i) := \begin{cases} x_i & \text{if } i \neq j \\ y_i & \text{if } i = j. \end{cases}$$

Then the function $g : I \to \bigcup_{i \in I} A_i$ is an element of the set A, and we have $\mathrm{pr}_j(g) = y_j$.

(b) The proof is as in (a): For an element $(y_j)_{j \in J}$ of the set $\prod_{j \in J} A_j$, let $(x_i)_{i \in I}$ be an arbitrary element of the direct product A, and define the function $g : I \to \bigcup_{i \in I} A_i$ by

$$g(i) := \begin{cases} x_i & \text{if } i \notin J \\ y_i & \text{if } i \in J. \end{cases}$$

Then the function $g : I \to \bigcup_{i \in I} A_i$ is an element of the set A, and we have $\mathrm{pr}_J(g) = (y_j)_{j \in J}$. □

6.3 Proposition. *Let I and J be two index sets, let $(I_j)_{j \in J}$ be a partition of the set I,*[a] *and let $(A_i)_{i \in I}$ be a family of sets. Define the function*

$$p : \prod_{i \in I} A_i \to \prod_{j \in J} \Big(\prod_{k \in I_J} A_k \Big) \text{ by } p : x = (x_i)_{i \in I} \mapsto \big(\mathrm{pr}_{I_j}(x)\big)_{j \in J}.$$

Then the function $p : \prod_{i \in I} A_i \to \prod_{j \in J} \big(\prod_{k \in I_J} A_k \big)$ is bijective.

[a] The family $(I_j)_{j \in J}$ is a partition of the set I if we have $\bigcup_{j \in J} I_j = I$ and if $I_j \cap I_k = \emptyset$ for all elements j and k of the set J such that $j \neq k$. For more details see Unit Direct Product [Garden 2020c].

Proof. Step 1. *The function $p : \prod_{i \in I} A_i \to \prod_{j \in J} \big(\prod_{k \in I_J} A_k \big)$ is injective:*

For, let $x = (x_i)$ and $y = (y_i)$ be two elements of the direct product $\prod_{i \in I} A_i$ such that $p(x) = p(y)$. By definition of the function p, it follows that

$$\big(\mathrm{pr}_{I_j}(x)\big)_{j \in J} = \big(\mathrm{pr}_{I_j}(y)\big)_{j \in J}, \text{ that is, } x_k = y_k \text{ for all } k \in I_j \text{ and all } j \in J.$$

Since $I = \bigcup_{j \in J} I_j$, we obtain $x_i = y_i$ for all elements i of the set I, that is, $x = y$.

Step 2. *The function $p : \prod_{i \in I} A_i \to \prod_{j \in J} \big(\prod_{k \in I_J} A_k \big)$ is surjective:*

For, let $y = (y_j)_{j \in J}$ be an element of the set $\prod_{j \in J} \big(\prod_{k \in I_J} A_k \big)$ with $y_j = (z_{j_k})_{k \in I_j}$. Let i be an element of the set i. Since $I = \bigcup_{j \in J} I_j$ (disjoint union), there exists exactly one element j of the set J such that the element i is contained in the set J. Hence, there exists exactly one element j_k of the set I_j such that $i = j_k$.

Define the function $x : I \to \bigcup_{i \in I} A_i$ by $x : i \mapsto z_i = z_{j_k}$. It follows that $p(x) = y$. □

Extensions of Functions:

6.4 Proposition. *Let I be an index set, and let $(A_i)_{i \in I}$ and $(B_i)_{i \in I}$ be two families of sets. For each element i of the set I, let $f_i : A_i \to B_i$ be a function from the set A_i into the set B_i.*

Define the function

$$f : A := \prod_{i \in I} A_i \to B := \prod_{i \in I} B_i$$

from the set A into the set B as follows:

Let $x = (x_i)_{i \in I}$ be an element of the set A. Set

$$f(x) := \big(f_i(x_i)\big)_{i \in I} \in B.$$

(a) If the functions $f_i : A_i \to B_i$ are injective for all elements i of the set I, then the function $f : A \to B$ is injective.

(b) If the functions $f_i : A_i \to B_i$ are surjective for all elements i of the set I, then the function $f : A \to B$ is surjective.

(c) If the functions $f_i : A_i \to B_i$ are bijective for all elements i of the set I, then the function $f : A \to B$ is bijective.

Proof. (a) Let $x = (x_i)$ and $y = (y_i)$ be two elements of the set A such that

$$\big(f_i(x_i)\big)_{i \in I} = f(x) = f(y) = \big(f_i(y_i)\big)_{i \in I}.$$

By Proposition 4.7, we have $f_i(x_i) = f_i(y_i)$ for all elements i of the set I. Since the functions $f_i : A_i \to B_i$ are injective, we get $x_i = y_i$ for all elements i of the set I. Again by Proposition 4.7, we get $x = y$.

(b) Let $b = (b_i)$ be an element of the set B. Since the functions $f_i : A_i \to B_i$ are surjective, for each element i of the set I, there exists an element a_i of the set A_i such that $f_i(a_i) = b_i$. Let $a := (a_i)_{i \in I}$. Then it follows that $f(a) = b$.

(c) follows from (a) and (b). □

7 Notes and References

We want to mention the books *The axiom of choice* by Thomas Jech [Jech 1973], *Axiom of Choice* by Horst Herrlich [Herrlich 2006] and *Consequences of the Axiom of Choice* by Paul Howard and Jean Rubin [Howard and Rubin 1998] which are devoted to the axiom of choice.

8 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Fraenkel, Adolf (1922). “Zu den Grundlagen der Cantor-Zermeloschen Mengenlehre”. In: *Mathematische Annalen* 86, pp. 230–237 (cit. on p. 10).

Herrlich, Horst (2006). *Axiom of Choice*. Lecture Notes in Mathematics. Heidelberg, Berlin, and New York: Springer Verlag (cit. on p. 23).

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Zermelo, Ernst (1904). “Beweis, dass jede Menge wohlgeordnet werden kann”. In: *Mathematische Annalen* 59, pp. 514–516 (cit. on pp. 20, 24).

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— (1967a). "Investigations in the Foundations of Set Theory I". In: *From Frege to Gödel. A Source Book in Mathematical Logic, 1879 - 1931*. Ed. by van Heijenoort, pp. 199–215. This article is a translation of [Zermelo 1908] into English. (Cit. on p. 21).

— (1967b). "Proof that every set can be well-ordered". In: *From Frege to Gödel. A Source Book in Mathematical Logic, 1879 - 1931*. Ed. by van Heijenoort, pp. 139–141. This article is a translation of [Zermelo 1904] into English. (Cit. on p. 20).

9 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *The Mathematical Universe*. Version 1.0.2. URL: https://www.math-garden.com/unit/nst-universe (cit. on pp. 3, 8, 10).

— (2020b). *Unions and Intersections of Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-unions (cit. on pp. 4, 8, 11, 12, 14).

— (2020c). *Direct Products and Relations*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-direct-products (cit. on pp. 16, 22).

— (2020d). *Functions and Equivalent Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-functions (cit. on pp. 3, 6, 7).

— (2020e). *Successor Sets and the Axioms of Peano*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-successor-sets (cit. on p. 8).

— (2020f). *Well-Ordered Sets*. Version 1.0.0. In preparation (cit. on p. 5).

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If you are willing to share comments and ideas to improve the present unit or hints about further references, we kindly ask you to send a mail to info@math-garden.com or to use the contact form on www.math-garden.com. Contributions are highly appreciated.

Index