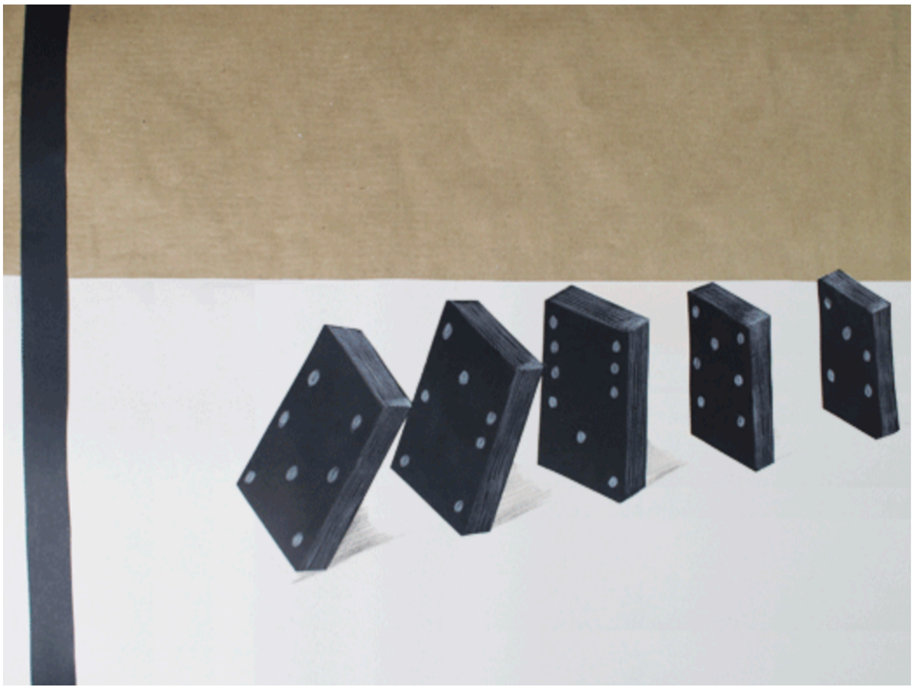
M. Garden

Successor Sets and The Axioms of Peano



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Title Page *Successor Sets and the Axioms of Peano*
Artwork by Gisela Naumann - www.malraum-bonn.de.

Version

Version 1.0.1 from August 2020

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Print

Printed with LaTeX.

Translations

For translations https://www.leo.org, https://www.dict.cc and https://translate.google.com have been used.

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Contents

1 Introduction

The present unit is part of the walk *The Axioms of Zermelo and Fraenkel*.

The most prominent system of axioms for mathematics is the axiomatics of Zermelo and Fraenkel abbreviated by ZFC where the letter C stands for the axiom of choice. It consists of the following axioms:

ZFC-0:	Basic Axiom
ZFC-1:	Axiom of Extension
ZFC-2:	Axiom of Existence
ZFC-3:	Axiom of Specification
ZFC-4:	Axiom of Foundation
ZFC-5:	Axiom of Pairing
ZFC-6:	Axiom of Unions
ZFC-7:	Axiom of Powers
ZFC-8:	Axiom of Substitution
ZFC-9:	Axiom of Choice
ZFC-10:	Axiom of Infinity

Axioms ZFC-0 to ZFC-4 are explained in Unit *The Mathematical Universe* [Garden 2020a]. Axioms ZFC-5 to ZFC-7 are explained in Unit *Unions and Intersections of Sets* [Garden 2020b]. Axioms ZFC-8 and ZFC-9 are explained in Unit *Families and the Axiom of Choice* [Garden 2020d]. Finally, Axiom ZFC-10 will be explained in the present unit.

I think it is really amazing, if not a small miracle, that ten axioms are a sufficient basis for (almost) the complete mathematical theory.

Successor Sets and the Axiom of Infinity (see Section 2):

Within the axiomatics of Zermelo and Fraenkel the natural numbers are defined as special sets. The main idea is as follows: One sets

$$\begin{aligned}
0 &:= \emptyset \\
1 &:= \{0\} = \{\emptyset\} = \emptyset \cup \{\emptyset\} = 0 \cup \{0\} \\
2 &:= \{0, 1\} = 1 \cup \{1\} \\
3 &:= \{0, 1, 2\} = 2 \cup \{2\} \\
&\vdots \\
n+1 &:= \{0, 1, \dots, n\} = n \cup \{n\}.
\end{aligned}$$

Then one defines the set $\mathbb{N}_0$ of the natural numbers as the set

$$\mathbb{N}_0 := \{0, 1, \dots, n, \dots\}.$$

The axiom of infinity will be needed to guarantee the existence of the set $\mathbb{N}_0$ of all natural numbers.

These definitions will only take place in Unit *Natural Numbers and the Principle of Induction* [Garden 2020e] introducing the natural numbers. The present unit is devoted to the necessary preparations consisting of the successor sets and the axioms of Peano.

Since the natural numbers will be defined recursively by

$$0 := \emptyset \text{ and } n+1 := n \cup \{n\},$$

the first step in this direction is the investigation of sets of the form

$$A \cup \{A\}.$$

Such a set is called a **successor set** (Definition 2.8), and it is denoted by

$$A^+ := A \cup \{A\}.$$

In Unit *Natural Numbers and the Principle of Induction* [Garden 2020e] we will define

$$n+1 := n^+ = n \cup \{n\},$$

that is, the number $n+1$ will be defined to be the successor of the number n. Successor sets have the following important properties:

(i) We have $A^+ \neq \emptyset$ (Proposition 2.3).

(ii) We have $A \in A^+$ and $A \subseteq A^+$ (Proposition 2.4).

(iii) We have $A^+ \neq A$ (Proposition 2.6).

The next question is how to construct the set

$$\mathbb{N}_0 = \{0, 1, 2, \ldots, n, n+1, \ldots\}$$

of all natural numbers: For its construction we need a new axiom, namely the axiom of infinity (Axiom 2.7) guaranteeing the existence of a set $\mathcal{A}$ fulfilling the following conditions:

(i) The empty set $\emptyset$ is an element of the set $\mathcal{A}$.

(ii) If the set A is an element of the set $\mathcal{A}$, then its successor A^+ is also an element of the set $\mathcal{A}$.

A set $\mathcal{A}$ fulfilling Conditions (i) and (ii) is called a **successor set** (Definition 2.8).

The most important conclusion of the axiom of infinity is the existence of a "minimal" successor set ω (see Theorem 2.10 and Definition 3.1). More precisely, Theorem 2.10 states that there exists exactly one successor set ω with the following property: If $\mathcal{S}$ is a successor set, then the set ω is a subset of the set $\mathcal{S}$.

The Minimal Successor Set (see Section 3):

As explained above the axiom of infinity will allow to define the set $\mathbb{N}_0$ of the natural numbers (see Unit *Natural Numbers and the Principle of Induction* [Garden 2020e]). The set $\mathbb{N}_0$ will be defined as the minimal successors set ω.

It follows from the definition of the minimal successor set ω that we have

$$0 = \emptyset \in \omega, 1 = 0^+ \in \omega, 2 = 1^+ \in \omega, \ldots, n+1 = n^+ \in \omega, \ldots$$

which motivates the definition $\mathbb{N}_0 := \omega$. Of course, we will define $n+1 := n^+$ for each natural number n.

The first important properties of the minimal successor set ω are as follows:

(i) Given a non-empty set A of the minimal successor set ω there exists a set B of the set ω such that $A = B^+$ (Theorem 3.4). This means: Given a natural number $n \neq 0$ there exists a natural number m such that $n = m + 1$.

(ii) Given two elements A and B of the minimal successor set ω such that $A^+ = B^+$ we have $A = B$ (Theorem 3.8). This means: Given two natural numbers m and n such that $m + 1 = n + 1$ we have $n = m$.

A technically important tool is the observation that the minimal successor set is a so-called transitive set (see Definition 3.6 and Proposition 3.7).

The Axioms of Peano (see Section 4):

The axioms of Peano have been introduced by Giuseppe Peano as an axiomatic basis for the definition of he natural numbers.

Within the axiomatics of Zermelo and Fraenkel we go one step further: We first give an axiomatic basis for the theory of sets or even for mathematics in general and then deduce the axioms of Peano from the axioms of Zermelo and Fraenkel.

The five axioms of Peano and their meaning are as follows (see Definition 4.1):

There exists a set ω such that

(P1) The set ω has a distinguished element 0. This means that the set $\mathbb{N}_0$ contains the number 0.

(P2) There exists a function $+ : \omega \to \omega$, $x \mapsto x^+$ from the set ω into itself. This means that the function $n \mapsto n + 1$ is a function from the set of the natural numbers into itself.

(P3) We have $x^+ \neq 0$ for all elements x of the set ω. This means that we have $n + 1 \neq 0$ for all natural numbers n.

(P4) If x and y are two elements of the set ω such that $x^+ = y^+$, then we have $x = y$. This means that $m + 1 = n + 1$ implies $m = n$ for all natural numbers m and n.

(P5) If A is a subset of the set ω such that

$$0 \in A \text{ and } x^+ \in A \text{ for all } x \in A,$$

then we have $A = \omega$. This means that if A is a subset of the set $\mathbb{N}_0$ such that

$$0 \in A \text{ and } n + 1 \in A \text{ for all } x \in A,$$

then we have $A = \mathbb{N}_0$.

A set X fulfilling the axioms of Peano is called a **Peano set** (Definition 4.1). Axiom (P5) guarantees that the set ω is not too big, and it allows to introduce the principle of induction. In the present unit we restrict ourselves to prove the fact that the minimal successor set fulfills the axioms of Peano (Theorem 4.2). Further conclusions like the principle of indction are subject of Unit *Natural Numbers and the Principle of Induction* [Garden 2020e]).

The Recursion Theorem (see Section 5):

Let us look at the recursive definition of the addition of two natural numbers m and n. We set

$$m + 0 := m \text{ and } m + (n + 1) := (m + n) + 1 \text{ for all } n \in \mathbb{N}_0.$$

In other words, we are looking for a function $\alpha_m : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\alpha_m(0) = m \text{ and } \alpha_m(n+1) = \alpha_m(n) + 1 \text{ for all } n \in \mathbb{N}_0.$$

Then we can define $m + n := \alpha_m(n)$.

More generally, let X be a set, let $f : X \to X$ be a function, and let x_0 be an element of the set X. Then we are looking for a function $\alpha : \mathbb{N}_0 \to X$ such that

$$\alpha(0) = x_0 \text{ and } \alpha(n+1) = f(\alpha(n)) \text{ for all } n \in \mathbb{N}_0.$$

In our example we have $X = \mathbb{N}_0$, $x_0 = m$ and $f(x) := x + 1$ for all elements x of the set $X = \mathbb{N}_0$. The existence of such a function $\alpha : \mathbb{N}_0 \to X$ is the content of the recursion theorem (Theorem 5.1).

Isomorphisms of Peano Sets (see Section 6):

As explained above one of the main results of this unit is the fact that the minimal successor set is a Peano set. In fact, up to isomorphisms, the minimal successor set is the *only* Peano set. More precisely, given a Peano set A there exists one (and only one) isomorphism $\alpha : \omega \to A$ from the minimal successor set ω onto the set A (Theorem 6.5).

2 Successor Sets and the Axiom of Infinity

Definition of a Successor:

2.1 Definition. Let A be a set. Then the set $A^+ := A \cup \{A\}$ is called the **successor** of the set A.

French / German. Successor = Successeur = Nachfolger.

2.2 Examples. (a) Let $A := \emptyset$ be the empty set. Then we have

$$A^+ = A \cup \{A\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}.$$

(b) Let $A := \{a, b\}$ for two elements a and b. Then we have

$$A^+ = A \cup \{A\} = \{a, b\} \cup \{\{a, b\}\} = \{a, b, \{a, b\}\}.$$

Elementary Properties of Successors:

2.3 Proposition. *Let A be a set, and let A^+ be its successor. Then the successor is non-empty.*

Proof. The assertion follows from the fact that the successor A^+ contains the set A as one of its elements. □

2.4 Proposition. *Let A be a set, and let A^+ be its successor. Then we have*

$$A \in A^+ \text{ and } A \subseteq A^+,$$

that is, the set A is at the same time an element and a subset of the set A^+.

Proof. The assertion follows from the definition of the successor $A^+ = A \cup \{A\}$. □

In the proof of Proposition 2.6 we will need the following result Unit *Unions and Intersections of Sets* [Garden 2020b]:

2.5 Theorem. *For each set A we have $A \notin A$.*

Proof. The theorem is a consequence of the axiom of foundation. For details see Unit *Unions and Intersections of Sets* [Garden 2020b]. □

2.6 Proposition. *Let A be a set, and let A^+ be its successor. Then we have $A^+ \neq A$.*

Proof. Assume that there exists a set A such that $A^+ = A$. It follows that $A = A^+ = A \cup \{A\}$. In particular, it follows that the set A is an element of itself, in contradiction to Theorem 2.5. □

The Axiom of Infinity:

2.7 Axiom. (ZFC-10: Axiom of Infinity) *There exists a set $\mathcal{A}$ fulfilling the following conditions:*

(i) The empty set $\emptyset$ is an element of the set $\mathcal{A}$.

(ii) If A is an element of the set $\mathcal{A}$, then its successor A^+ is also an element of the set $\mathcal{A}$.

French / German. Axiom of Infinity = Axiome de l'infini = Unendlichkeitsaxiom

Definition of a Successor Set:

2.8 Definition. Let $\mathcal{A}$ be a set fulfilling the following conditions:

(i) The empty set $\emptyset$ is an element of the set $\mathcal{A}$.

(ii) If A is an element of the set $\mathcal{A}$, then its successor A^+ is also an element of the set $\mathcal{A}$.

Then the set $\mathcal{A}$ is called a **successor set**.

Note that the axiom of infinity guarantees that there exists at least one successor set.

Elementary Properties of Successor Sets:

2.9 Proposition. *Let $\mathcal{C}$ be a non-empty set of successor sets. Then the intersection*

$$\mathcal{I} := \bigcap_{\mathcal{A} \in \mathcal{C}} \mathcal{A}$$

is also a successor set.

Proof. Step 1. *The empty set $\emptyset$ is an element of the set $\mathcal{I}$:*

Since every set $\mathcal{A}$ of the set $\mathcal{C}$ is a successor set, each of these sets contains the empty set. It follows that the set $\mathcal{I} := \bigcap_{\mathcal{A}\in\mathcal{C}} \mathcal{A}$ also contains the empty set.

Step 2. *If a set A is an element of the set $\mathcal{I}$, then its successor A^+ is also an element of the set $\mathcal{I}$:*

For, let A be an element of the set $\mathcal{I}$. It follows that the set A is an element of every set $\mathcal{A}$ of the set $\mathcal{C}$. Since every set $\mathcal{A}$ of the set $\mathcal{C}$ is a successor set, each of these sets contains the successor A^+ of the set A. It follows that the set $\mathcal{I} := \bigcap_{\mathcal{A}\in\mathcal{C}} \mathcal{A}$ also contains the successor A^+. □

2.10 Theorem. *There exists exactly one successor set ω with the following property: If $\mathcal{S}$ is a successor set, then the set ω is a subset of the set $\mathcal{S}$.*

Proof. Step 1. *Existence:*

Step 1.1. *Definition of the set ω:*

By the axiom of infinity (Axiom 2.7), there exists a successor set $\mathcal{A}$. Let

$$\mathcal{C} := \{\mathcal{X} \subseteq \mathcal{A} \mid \mathcal{X} \text{ is a successor set}\}$$

be the set of the subsets of the successor set $\mathcal{A}$ which are themselves successor sets. Since the set $\mathcal{A}$ is an element of the set $\mathcal{C}$, we have $\mathcal{C} \neq \emptyset$. Set

$$\omega := \bigcap_{\mathcal{X}\in\mathcal{C}} \mathcal{X}.$$

Step 1.2. *The set ω is a successor set:*

The assertion follows from Proposition 2.9.

Step 1.3. *Let $\mathcal{S}$ be a successor set. Then the set ω is a subset of the set $\mathcal{S}$:*

By Proposition 2.9, the set $\mathcal{D} := \mathcal{S} \cap \mathcal{A}$ is an element of the set

$$\mathcal{C} = \{\mathcal{X} \subseteq \mathcal{A} \mid \mathcal{X} \text{ is a successor set}\}$$

defined in Step 1.1. It follows that

$$\omega = \bigcap_{\mathcal{X}\in\mathcal{C}} \mathcal{X} \subseteq \mathcal{D} = \mathcal{S} \cap \mathcal{A} \subseteq \mathcal{S}.$$

Step 2. *Uniqueness:*

Let ω and ω' be two successor sets with the property: If $\mathcal{S}$ is a successor set, then the set ω and the set ω' are subsets of the set $\mathcal{S}$.

Since the sets ω and ω' are successor sets, it follows that

$$\omega \subseteq \omega' \text{ and } \omega' \subseteq \omega$$

implying that $\omega = \omega'$. □

2.11 Remark. The proof of Theorem 2.10 would have been easier with a definition of the form

$$\omega := \bigcap \{\mathcal{X} \mid \mathcal{X} \text{ is a successor set}\}.$$

But intersections are only defined if they are of the form

$$\omega := \bigcap \{\mathcal{X} \in \mathcal{C} \mid \mathcal{X} \ldots\}$$

for some set $\mathcal{C}$ (for more details see Unit *Unions and Intersections of Sets* [Garden 2020b]).

3 The Minimal Successor Set

Definition of the Minimal Successor Set:

3.1 Definition. Let ω be the successor set with the following property: If $\mathcal{S}$ is a successor set, then the set ω is a subset of the set $\mathcal{S}$ (Theorem 2.10).

The successor set ω is called the **minimal successor set**. It is denoted by ω.

3.2 Remark. In Unit *Natural Numbers and the Principle of Induction* [Garden 2020e] we will set $\mathbb{N}_0 := \omega$, that is, we will define that the set of the natural numbers is the minimal successor set ω. In addition, we will set $n + 1 := n^+$.

Elementary Properties of the Minimal Successor Set:

3.3 Theorem. *Let ω be the minimal successor set, and let $\mathcal{S}$ be a successor set which is a subset of the set ω. Then we have $\mathcal{S} = \omega$.*

Proof. By Theorem 2.10, the set ω is a subset of the set $\mathcal{S}$. It follows from

$$\omega \subseteq \mathcal{S} \text{ and } \mathcal{S} \subseteq \omega$$

that $\mathcal{S} = \omega$. □

3.4 Theorem. *Let ω be the minimal successor set, and let $\emptyset \neq A$ be an element of the set ω. Then there exists an element B of the set ω such that $A = B^+$.*

Proof. Let

$$\mathcal{S} := \{\emptyset\} \cup \{A \in \omega \mid \exists\, Z \in \omega \text{ such that } A = Z^+\}.$$

By definition of the set $\mathcal{S}$, the empty set is an element of the set $\mathcal{S}$. If A is an element of the set $\mathcal{S}$, then the set $B := A^+$ is obviously an element of the set $\mathcal{S}$. It follows from Theorem 3.3 that $\mathcal{S} = \omega$. □

3.5 Remark. Theorem 3.4 is an important technical tool in the study of successor sets. In view of the definition $\mathbb{N} = \omega$ it just means that for each natural number $n \neq 0$ there exists a natural number m such that $n = m + 1$.

3.6 Definition. A set A is called a **transitive set** if every element of the set A is at the same time a subset of the set A.

French / German. Transitive set = Ensemble transitif = Transitive Menge.

Proposition 3.7 will be used in the proof of Theorem 3.8.

3.7 Proposition. *Let ω be the minimal successor set.*

(a) The minimal successor set ω is a transitive set.

(b) Let A be an element of the set ω. Then the set A is a transitive set.

Proof. (a) Let
$$\mathcal{S} := \{A \in \omega \mid A \subseteq \omega\}.$$
In oder to show that the minimal successor set ω is a transitive set, we have to show that $\mathcal{S} = \omega$.

For, note that the empty set is an element of the set $\mathcal{S}$. Suppose that A is an element of the set $\mathcal{S}$. By definition of the set $\mathcal{S}$, the element A is an element and a subset of the minimal successor set ω. It follows that
$$A^+ = A \cup \{A\} \subseteq \omega, \text{ that is, } A^+ \in \mathcal{S}.$$
It follows that the set $\mathcal{S}$ is a successor set. By Theorem 3.3, we obtain $\mathcal{S} = \omega$.

(b) Step 1. *Definition of the set $\mathcal{S}$:*

Let
$$\mathcal{S} := \{A \in \omega \mid A \text{ is transitive}\}.$$
In oder to show that every element A of the minimal successor set ω is a transitive set, we have to show that $\mathcal{S} = \omega$.

Step 2. *The set $\mathcal{S}$ is a successor set:*

Step 2.1. *The empty set is an element of the set $\mathcal{S}$:*

Assume that the empty set $A := \emptyset$ is no element of the set $\mathcal{S}$. Then there exists an element X of the set A which is not transitive, in contradiction to the fact that the set $A = \emptyset$ does not contain any element.

Step 2.2. *Let A be an element of the set $\mathcal{S}$. Then the successor A^+ of the set A is also an element of the set $\mathcal{S}$:*

Let A be an element of the set $\mathcal{S}$, and let X be an element of the set $A^+ = A \cup \{A\}$.

Case 1. Suppose that the element X is an element of the set A. Since the set A is transitive, it follows that the set X is a subset of the set A. Hence, we have
$$X \subseteq A \subseteq A^+ \text{ (Proposition 2.4)}.$$
Case 2. Suppose that the element X is an element of the set $\{A\}$, that is, $X = A$. It follows that
$$X = A \subseteq A^+ \text{ (Proposition 2.4)}.$$
Step 3. *We have $\mathcal{S} = \omega$:*

By Step 2, the set $\mathcal{S}$ is a successor set which is a subset of the set ω. It follows from Theorem 3.3 that we have $\mathcal{S} = \omega$. □

3.8 Theorem. *Let ω be the minimal successor set, and let A and B be two elements of the set ω. If $A^+ = B^+$, then we have $A = B$.*

Proof. Let A and B be two elements of the set ω such that $A^+ = B^+$. Since we have

$$A \in A \cup \{A\} = A^+ = B^+,$$

the set A is an element of the set $B^+ = B \cup \{B\}$. It follows that the set A is contained in the set B or that $A = B$.

If $A = B$, the proof is finished. Hence, we may suppose that the set A is an element of the set B. It follows from Proposition 3.7 that the set A is a subset of the set B.

In the same way, we see that $B = A$ or that the set B is a subset of the set A. It follows from

$$A \subseteq B \text{ and } B \subseteq A$$

that $A = B$. □

Historical Notes:

The idea of an infinite set is quite old. Early definitions have probably been of the following form: A set A is called infinite if we have

$$|A| > n \text{ for all } n \in \mathbb{N}.$$

A good example for this approach is the wording of Euclid expressing that there exist infinitely many prime numbers:

Proposition 20. *Prime numbers are more than any assigned multitude of prime numbers.*

See [Heath 1908, Book IX, no. 20, p. 412].

In the context of the axiomatization of mathematics it was a new idea that one has to give an argument why infinite sets, in particular the set $\mathbb{N}_0$ of the natural numbers, exist.

The first breakthrough was the paper *Was sind und was sollen die Zahlen?* by Richard Dedekind [Dedekind 1888]. In this paper he gives a formal introduction of the natural numbers and gives a formal definition of an infinite set:

64. Erklärung. *Ein System S heißt* **unendlich**, *wenn es einem echten Teile seiner selbst ähnlich ist [...]; im entgegengesetzten Fall heißt S ein* **endliches** *System.*

See [Dedekind 1932, Vol. 1, p. 356].

64. Explanation. *A system S is called* **infinite** *if it is similar to a proper subset of the system S [...]; in the opposite case S is called a* **finite** *system.*

(Translation by the author.)

Dedekind uses the word system for sets. Two sets A and B are called similar if there exists a bijective mapping from the set A onto the set B.

Dedekind also postulated the existence of an infinite set:

66. Satz. *Es gibt unendliche Systeme.*

See [Dedekind 1932, Vol. 1, p. 357].

66. Theorem. *There exist infinite systems.*

(Translation by the author.)

Dedekind's proof is of a more philosophical nature and cannot be seen as a rigorous mathematical proof. If you want, you can understand his "proof" as a (philosophical) argument that makes the postulation of the existence of infinite sets more plausible. With this interpretation one may say that Dedekind was the first to introduce the axiom of infinity.

By the way, the further procedure of Dedekind is quite similar to the procedure explained above: The axiom of infinity provides the existence of at least one successor set. As a next step the minimal successor set ω is constructed followed by the definition $\mathbb{N}_0 := \omega$.

Dedekind also postulates the existence of an infinite set and then constructs a specific infinite set N followed by the definition $\mathbb{N}_0 := N$.

Ernst Zermelo presents his first version of the axiom of infinity in his first axiomatization paper [Zermelo 1908] as follows:

Axiom VII (Axiom des Unendlichen). *Der Bereich enthält mindestens eine Menge* Z*, welche die Nullmenge als Element enthält und so beschaffen ist, dass jedem ihrer Elemente* a *ein weiteres Element der Form* $\{a\}$ *entspricht, oder welche mit jedem ihrer Elemente* a *auch die entsprechende Menge* $\{a\}$ *enthält.*

See [Zermelo 1908, pp. 266-267].

Axiom VII (Axiom of Infinity). *There exists in the domain at least one set* Z *that contains the null set as an element and is so constituted that to each of its elements* a *there corresponds a further element of the form* $\{a\}$*, in other words, that with each of its elements* a *it also contains the corresponding set* $\{a\}$ *as an element.*

See [Zermelo 1967, p. 204].

The domain is the mathematical universe (see Unit *The Mathematical Universe* [Garden 2020a]). This axiom is very close to our axiom 2.7, but there is an important difference:

Zermelo has in mind to construct the natural numbers as follows:

$$0 := \emptyset,\ 1 := \{0\} = \{\emptyset\},\ 2 := \{1\} = \{\{\emptyset\}\},\ 3 := \{2\} = \{\{\{\emptyset\}\}\}, \dots$$

During the investigation of ordinal numbers (see Unit *Ordinal Numbers* [**nst-ordinal-numbers**]), John von Neumann introduced the following way to define natural numbers:

$$0 := \emptyset,\ 1 := 0 \cup \{0\} = \{0\} = \{\emptyset\},\ 2 := 1 \cup \{1\} = \{0, 1\} = \{\emptyset, \{\emptyset\}\},$$

$$3 := 2 \cup \{2\} = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}, \dots$$

In [von Neumann 1922] this reads as follows:

$$\begin{aligned} 0 &:= 0, \\ 1 &:= (0), \\ 2 &:= (0, (0)), \\ 3 &:= (0, (0), (0, (0))). \end{aligned}$$

See [von Neumann 1922, p. 199].

4 The Axioms of Peano

The Axioms of Peano:

4.1 Definition. Let A be a set.

(a) The set A **fulfills the axioms of Peano** if it fulfills the following conditions:

(P1) The set A contains a distinguished element 0. In particular, the set A is not empty.

(P2) There exists a function $^+ : A \to A$, $x \mapsto x^+$ from the set A into itself.

(P3) We have $x^+ \neq 0$ for all elements x of the set A.

(P4) If x and y are two elements of the set A such that $x^+ = y^+$, then we have $x = y$, that is, the function $^+ : A \to A$ is injective.

(P5) If B is a subset of the set A such that

$$0 \in B \text{ and } x^+ \in B \text{ for all } x \in B,$$

then we have $B = A$.

(b) Let A be a set fulfilling the axioms of Peano. Then the set A is called a **Peano set**.

French / German. Peano set = Ensemble de Peano = Peano-Menge.

4.2 Theorem. *The minimal successor set ω is a Peano set where $0 := \emptyset$ and $A^+ := A \cup \{A\}$ for all elements A of the set ω.*

Proof. (P1) follows from the definition of a successor set (Definition 2.8).

(P2) Recall that a function $f : \omega \to \omega$ is a subset F of the direct product $\omega \times \omega$ such that for each element A of the set ω there exists exactly one element B of the set ω such that the pair (A, B) is contained in the set F. In this case we write $B = f(A)$. For more details see Unit *Functions and Equivalent Sets* [Garden 2020c].

Set

$$F := \{(A, A^+) \mid A \in \omega\}.$$

The set F defines a function $f : \omega \to \omega$ such that

$$f(A) = A^+ \text{ for all } A \in \omega.$$

(P3) follows from Proposition 2.3.

(P4) follows from Theorem 3.8.

(P5) follows from Theorem 3.3. □

Elementary Properties of Peano Sets:

4.3 Proposition. *Let A be a set fulfilling the axioms of Peano, and let $0 \neq x$ be an element of the set A. Then there exists an element y of the set A such that $x = y^+$.*

Proof. Let

$$B := \{0\} \cup \{x \in A \mid \exists\, y \in A \text{ such that } x = y^+\}.$$

By definition of the set B, it contains the element 0. If x is an element of the set B, then the element x^+ is obviously contained in the set B. It follows from Axiom (P5) that $B = A$. □

4.4 Proposition. *Let A be a set fulfilling the axioms of Peano. Then the function $\alpha : A \to A \setminus \{0\}$, $\alpha : x \mapsto x^+$ is bijective.*

Proof. It follows from Axiom (P4) of the axioms of Peano (Definition 4.1) that the function $\alpha : A \to A \setminus \{0\}$ is injective. It follows from Proposition 4.3 that the function $\alpha : A \to A \setminus \{0\}$ is surjective. $\square$

Historical Notes:

The axioms of Peano have been introduced by Giuseppe Peano:

Axiomata

1. **1 ε N.**
2. ***a* ε N . Ɔ . *a* = *a*.**
3. ***a*, *b*, *c* ε N . Ɔ : *a* = *b* . = . *b* = *a*.**
4. ***a*, *b* ε N . Ɔ ∴ *a* = *b* . *b* = *c* : Ɔ . *a* = *c*.**
5. ***a* = *b* . *b* ε N : Ɔ . *a* ε N.**
6. ***a* ε N . Ɔ . *a* + 1 ε N.**
7. ***a*, *b* ε N . Ɔ : *a* = *b* . = . *a* + 1 = *b* + 1.**
8. ***a* ε N . Ɔ . *a* + 1 −= 1.**
9. ***k* ε K ∴ 1 ε *k* ∴ *x* ε N . *x* ε *k* : Ɔ$_x$. *x* + 1 ε *k* :: Ɔ . N Ɔ *k*.**

1. $1 \in N$.

2. $a \in N \Rightarrow a = a$.

3. $a, b, c \in N \Rightarrow (a = b \Leftrightarrow b = a)$.

4. $a, b \in N \Rightarrow \big((a = b) \wedge (b = c) \Rightarrow (a = c)\big)$.

5. $(a = b) \wedge (b \in N) \Rightarrow a \in N$.

6. $a \in N \Rightarrow a + 1 \in N$.

7. $a, b \in N \Rightarrow \big((a = b) \Leftrightarrow (a + 1 = b + 1)\big)$.

8. $a \in N \Rightarrow a + 1 \neq 1$.

9. $(1 \in K) \wedge \big((x \in K) \Rightarrow (x + 1 \in K)\big) \Rightarrow N \subseteq K$.

Definitiones

10. $2 = 1 + 1$; $3 = 2 + 1$; $4 = 3 + 1$; *etc.*

See [Peano 1889, p. 1].

We have transformed the original text of Peano into modern notations. Condition (1) corresponds to Condition (P1) of Definition 4.1. Peano starts with the number 1, whereas we have started with the number 0. Condition (6) corresponds to Condition (P2). Condition (8) corresponds to Condition (P3). Condition (7) corresponds to Condition (P4). Finally, Condition (9) corresponds to Condition (P5).

5 The Recursion Theorem

The recursion theorem allows the recursive definition of functions. We will apply it in Theorem 6.4.

5.1 Theorem. (Recursion Theorem) *Let X be a non-empty set, let $f : X \longrightarrow X$ be a function from the set X into itself, and let a be an element of the set X.*

Then there exists exactly one function $\alpha : \omega \longrightarrow X$ from the minimal successor set ω into the set X fulfilling the following conditions:

(i) We have $\alpha(\emptyset) = a$.

(ii) We have $\alpha(A^+) = f(\alpha(A))$ for each element A of the set ω.

French / German. Recursion theorem = Thórème de récursivité = Rekursionssatz.

Proof. Note that a function $\alpha : \omega \longrightarrow X$ is a subset F_α of the direct product $\omega \times X$ such that for each element A of the set ω, there exists exactly one element x of the set X such that the pair (A, x) is contained in the set F_α. In this case, we write $\alpha(A) = x$ (for more details see Unit *Functions and Equivalent Sets* [Garden 2020c]).

We shall construct the set F_α explicitly:

Step 1. *Definition of the set F_α:*

Let $\mathcal{F}$ be the set of all subsets F of the direct product $\omega \times X$ fulfilling the following conditions:

(i) We have $(\emptyset, a) \in F$.

(ii) If the pair (A, x) is contained in the set F, then the pair $(A^+, f(x))$ is also contained in the set F.

Obviously, the set $\omega \times X$ fulfills Conditions (i) and (ii) implying that $\mathcal{F} \neq \emptyset$. We set

$$F_\alpha := \bigcap_{F \in \mathcal{F}} F.$$

Step 2. *The set F_α fulfills the Conditions (i) and (ii) of Step 1:*

The assertion follows from the fact that each element F of the set $\mathcal{F}$ fulfills the Conditions (i) and (ii) and that $F_\alpha = \bigcap_{F \in \mathcal{F}} F$.

Step 3. *The set F_α is a function:*

For, let

$$\mathcal{S} := \{A \in \omega \mid \text{ there exists exactly one element } x \in X \text{ such that } (A, x) \in F_\alpha\}.$$

We have to show that we have $\mathcal{S} = \omega$:

Step 3.1. *The empty set $\emptyset$ is an element of the set $\mathcal{S}$:*

We have to show that there exists exactly one element x of the set X such that the pair $(\emptyset, x)$ is contained in the set F_α:

Existence: By definition of the set F_α, the pair $(\emptyset, a)$ is contained in the set F_α.

Uniqueness: Assume that there exists an element $b \neq a$ of the set X such that the pair $(\emptyset, b)$ is contained in the set F_α. Let $F'_\alpha := F_\alpha \setminus \{(\emptyset, b)\}$. We claim that the set F'_α is an element of the set $\mathcal{F}$ defined in Step 1:

Since $b \neq a$ and since the pair $(\emptyset, a)$ is contained in the set F_α, it follows that the pair $(\emptyset, a)$ is also contained in the set F'_α.

Let (A, x) be an element of the set F'_α. Since the set F'_α is a subset of the set F_α, the pair (A, x) is contained in the set F_α. It follows from Step 2 that the pair $(A^+, f(x))$ is contained in the set F_α. Since $A^+ \neq \emptyset$ (Proposition 2.3), it follows that the pair $(A^+, f(x))$ is also contained in the set $F'_\alpha = F_\alpha \setminus \{(\emptyset, b)\}$.

Hence, the set F'_α is an element of the set $\mathcal{F}$, and it follows that

$$F_\alpha = \bigcap_{F \in \mathcal{F}} F \subseteq F'_\alpha = F_\alpha \setminus \{(\emptyset, b)\} \subset F_\alpha,$$

a contradiction.

Step 3.2. *Let* A *be an element of the set* $\mathcal{S}$*. Then the successor* A^+ *is also an element of the set* $\mathcal{S}$*:*

We have to show that there exists exactly one element z of the set X such that the pair (A^+, z) is contained in the set F_α:

Existence: Since the set A is an element of the set $\mathcal{S}$, there exists a (unique) element x of the set X such that the pair (A, x) is contained in the set F_α. By Step 2, the set F_α fulfills Condition (ii) of Step 1 implying that the pair $(A^+, f(x))$ is contained in the set F_α.

Uniqueness: Assume that there exists an element A of the set $\mathcal{S}$ such that there exist two distinct elements b and c of the set X such that the pairs (A^+, b) and (A^+, c) are contained in the set F_α. Since the set A is an element of the set $\mathcal{S}$, there exists a unique element x of the set X such that the pair (A, x) is contained in the set F_α.

Since, by Step 2, the set F_α fulfills Condition (ii) of Step 1, the pair $(A^+, f(x))$ is contained in the set F_α. W.l.o.g. suppose that $c \neq f(x)$.

Let $F'_\alpha := F_\alpha \setminus \{(A^+, c)\}$. We claim that the set F'_α is an element of the set $\mathcal{F}$:

Since $A^+ \neq \emptyset$, it follows that the pair $(\emptyset, a)$ is contained in the set $F'_\alpha = F_\alpha \setminus \{(A^+, c)\}$.

Let (B, y) be an element of the set F'_α. If $B = A$, then the pair $(B^+, f(x)) = (A^+, f(x))$ is contained in the set F'_α.

If $B \neq A$, we have $B^+ \neq A^+$ (Theorem 3.8). It follows from Step 2 that the pair $(B^+, f(y))$ is contained in the set F_α. Since $B^+ \neq A^+$, pair $(B^+, f(y))$ is also contained in the set $F'_\alpha = F_\alpha \setminus \{(A^+, c)\}$.

Hence, the set F'_α is an element of the set $\mathcal{F}$, and it follows that

$$F_\alpha = \bigcap_{F \in \mathcal{F}} F \subseteq F'_\alpha = F_\alpha \setminus \{(A^+, c)\} \subset F_\alpha,$$

a contradiction.

Step 3.3. *We have* $\mathcal{S} = \omega$*:*

By Step 3.1 and 3.2, the set $\mathcal{S}$ is a successor set which is a subset of the set ω. It follows from Theorem 3.3 that $\mathcal{S} = \omega$. Hence, the set F_α is a function.

Step 4. *The set* F_α *is a function from the set* ω *into the set* X *such that* $\alpha(\emptyset) = a$ *and* $\alpha(A^+) = f(\alpha(A))$ *for each element* A *of the set* ω*:*

By Step 3, the set $F_\alpha \subseteq \omega \times X$ is a function which we denote by $\alpha : W \to X$.

By Step 2, the set F_α contains the pair $(\emptyset, a)$ implying that $\alpha(\emptyset) = a$.

Let A be an element of the set ω, and let x be the unique element of the set X such that $(A, x) \in F_\alpha$, that is, $\alpha(A) = x$. By Step 2, it follows that the pair $(A^+, f(x))$ is contained in the set F_α. It follows that

$$\alpha(A^+) = f(x) = f(\alpha(A)).$$

Step 5. *Let $\alpha : \omega \to X$ and $\beta : \omega \to X$ be two functions from the set ω into the set X fulfilling the conditions*

(i) $\alpha(\emptyset) = a$ and $\beta(\emptyset) = a$.

(ii) $\alpha(A^+) = f\big(\alpha(A)\big)$ and $\beta(A^+) = f\big(\beta(A)\big)$.

Then we have $\alpha = \beta$:

Let $S := \{A \in \omega \mid \alpha(A) = \beta(A)\}$. We have to show that $S = \omega$:

Since $\alpha(\emptyset) = a = \beta(\emptyset)$, the empty set is an element of the set S. Let A be an element of the set S. Then it follows that

$$\alpha(A^+) = f\big(\alpha(A)\big) = f\big(\beta(A)\big) = \beta(A^+),$$

that is, the set A^+ is an element of the set S. By Theorem 3.3, we have $S = \omega$. □

Historical Notes:

The method of the recursive definition of a sequence is quite old. For example, Blaise Pascal uses a recursive definition in his *Traité du Triangle Arithmétique* from 1665 in order to define what is today called the binomial coefficients:

Le nombre de chaque cellule est égal à celui de la cellule qui la précède dans son rang perpendiculaire, plus à celui qui la précède dans son rang parallèle.

See [Pascal 1665, Traité du Triangle, p. 1].

The number in each cell is equal to the number in the cell which precedes it in its perpendicular rang plus the number in the cell which precedes it in its parallel rang.

(Translation by the author.)

In modern terminology Pascal defines the binomial coefficients by the equation

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

For more details about binomial coefficients and the Triangle of Pascal see Unit *The Binomial Coefficients and the Triangle of Pascal* [Garden 2020f].

The formal definition and proof of the recursion theorem has been given by Richard Dedekind in 1888:

126. Satz. *Ist eine beliebige (ähnliche oder unähnliche) Abbildung θ eines Systems Ω in sich selbst und außerdem ein bestimmtes Element ω aus Ω gegeben, so gibt es eine und nur eine Abbildung ψ der Zahlenreihe N, welche den Bedingungen*

I. $\psi(N) \subseteq \Omega$,

II. $\psi(1) = \omega$,

III. $\psi(n') = \theta\psi(n)$ genügt, wo n jede Zahl bedeutet.

See [Dedekind 1932, Volume 1, p. 371].

126. Theorem. *If θ is a (similar or non-similar) mapping from a system Ω into itself and if there is an element ω of Ω, then there exists one and only one mapping ψ from the number line N which fulfills the conditions*

I. $\psi(N) \subseteq \Omega$,

II. $\psi(1) = \omega$,

III. $\psi(n') = \theta\psi(n)$ *where* n *is any number.*

(Translation by the author.)

Similar means bijective. The number line N is of course the set $\mathbb{N}_0$ of the natural numbers. The expression $\theta\psi(n)$ means $\theta(\psi(n))$. Furthermore, we have $n' = n^+ = n + 1$.

6 Isomorphisms of Peano Sets

6.1 Definition. Let A and B be two Peano sets.

(a) A function $\alpha : A \to B$ is called an **isomorphism from the Peano set A onto the Peano set B** if the following conditions are fulfilled:

(i) The function $\alpha : A \to B$ is bijective. Let $\beta := \alpha^{-1} : B \to A$.

(ii) We have $\alpha(0_A) = 0_B$.

(iii) We have $\alpha(x^+) = \alpha(x)^+$ for all elements x of the set A.

(iv) We have $\beta(0_B) = 0_A$.

(v) We have $\beta(y^+) = \beta(y)^+$ for all elements y of the set B.

(b) If A and B are two Peano sets such that there exists an isomorphism $\alpha : A \to B$ from the Peano set A onto the Peano set B, then the Peano sets A and B are called **isomorphic**. In this case we write $A \cong B$.

French / German. Isomorphic Peano sets = Ensembles de Peano isomorphes = Isomorphe Peano-Mengen.

6.2 Proposition. *Let A and B be two Peano sets.*

(a) Let $\alpha : A \to B$ be an isomorphism from the Peano set A onto the Peano set B, and let $\beta := \alpha^{-1} : B \to A$. Then the function $\beta : B \to A$ is an isomorphism from the Peano set B onto the Peano set A.

(b) Let $\alpha : A \to B$ be a bijective function from the Peano set A onto the Peano set B such that

$$\alpha(0_A) = 0_B \text{ and } \alpha(x^+) = \alpha(x)^+ \text{ for all } x \in A.$$

Then the function $\alpha : A \to B$ is an isomorphism from the Peano set A onto the Peano set B.

Proof. (a) The proof is obvious.

(b) Let $\beta := \alpha^{-1} : B \to A$. Since $\alpha(0_A) = 0_B$, we have $\beta(0_B) = 0_A$.

Let y be an element of the set B. Then we have

$$\alpha\big(\beta(y^+)\big) = y^+ \text{ and } \alpha\big(\beta(y)^+\big) = \alpha\big(\beta(y)\big)^+ = y^+.$$

Since the function $\alpha : A \to B$ is injective, it follows that $\beta(y^+) = \beta(y)^+$. $\square$

6.3 Proposition. *Let A, B and C be three Peano sets, and let $\alpha : A \to B$ and $\beta : B \to C$ be two isomorphisms from the Peano set A onto the Peano set B and from the Peano set B onto the Peano set C, respectively.*

Then the composite $\gamma := \beta \circ \alpha : A \to C$ is an isomorphism from the Peano set A onto the Peano set C.

Proof. The proof is obvious. □

6.4 Theorem. *Let A be a Peano set, and let ω be the minimal successor set. Then the Peano sets A and ω are isomorphic.*

Proof. Step 1. *Definition of the function $\alpha : \omega \to A$:*

By Axiom (P2), the function $f : A \to A$, $x \mapsto x^+$ exists. It follows from the recursion theorem (Theorem 5.1) that there exists a function $\alpha : \omega \to A$ such that

$$\alpha(\emptyset) = 0 \text{ and } \alpha(X^+) = f(\alpha(X)) = \alpha(X)^+ \text{ for all } X \in \omega.$$

Step 2. *The function $\alpha : \omega \to A$ is injective:*

Let

$$\mathcal{S} := \{X \in \omega \mid \alpha(X) \neq \alpha(Y) \text{ for all } Y \in \omega \text{ with } Y \neq X\}.$$

Step 2.1. *The empty set is an element of the set $\mathcal{S}$:*

Assume that there exists an element X of the set ω such that $X \neq \emptyset$ and $\alpha(X) = \alpha(\emptyset)$. Since $X \neq \emptyset$, it follows from Theorem 3.4 that there exists an element Y of the set ω such that $Y^+ = X$. Let $y := \alpha(Y)$. Then we have

$$y^+ = \alpha(Y)^+ = \alpha(Y^+) = \alpha(X) = \alpha(\emptyset) = 0,$$

in contradiction to Axiom (P3).

Step 2.2. *Let X be an element of the set $\mathcal{S}$. Then the successor X^+ of the set X is also an element of the set $\mathcal{S}$:*

Assume that there exists an element X of the set $\mathcal{S}$ such that the successor X^+ of the set X is no element of the set $\mathcal{S}$. Then there exists an element Y of the set ω such that $Y \neq X^+$ and $\alpha(X^+) = \alpha(Y)$.

We claim that $Y \neq \emptyset$: Since $X^+ \neq \emptyset$ (Proposition 2.3) and since the empty set is an element of the set $\mathcal{S}$ (Step 2.1), we have $\alpha(X^+) \neq \alpha(\emptyset)$. Assume that $Y = \emptyset$. Then it follows that

$$\alpha(X^+) = \alpha(Y) = \alpha(\emptyset) \neq \alpha(X^+),$$

a contradiction.

It follows from Theorem 3.4 that there exists an element Z of the set ω such that $Z^+ = Y$. Hence, we have

$$\alpha(X)^+ = \alpha(X^+) = \alpha(Y) = \alpha(Z^+) = \alpha(Z)^+.$$

It follows from Axiom (P4) that $\alpha(X) = \alpha(Z)$. Since the set X is an element of the set $\mathcal{S}$, we have $X = Z$. It follows that

$$X^+ = Z^+ = Y,$$

in contradiction to the fact that $Y \neq X^+$.

Step 2.3. *The function* $\alpha : \omega \to A$ *is injective:*

Since the set $\mathcal{S}$ is a successor set (Step 2.1 and 2.2) which is a subset of the minimal successor set ω, it follows from Theorem 3.3 that we have $\mathcal{S} = \omega$. Hence, the function $\alpha : \omega \to A$ is injective.

Step 3. *The function* $\alpha : \omega \to A$ *is surjective:*

Let

$$B := \{x \in A \mid \exists\, X \in \omega \text{ such that } \alpha(X) = x\} = \mathrm{Image}(\alpha).$$

Since $\alpha(\emptyset) = 0$, the set B contains the element 0. Let y be an element of the set B. Then there exists an element X of the set ω such that $\alpha(X) = y$. It follows from

$$\alpha(X^+) = \alpha(X)^+ = y^+$$

that the element y^+ is also contained in the set B. By Axiom (P5), we have $B = A$, that is, the function $\alpha : \omega \to A$ is surjective.

Step 4. *The mapping* $\alpha : \omega \to X$ *is an isomorphism from the Peano set* ω *onto the Peano set* A*:*

By Step 2 and Step 3, the function $\alpha : \omega \to X$ is bijective. By definition of the function $\alpha : \omega \to X$, we have

$$\alpha(\emptyset) = 0 \text{ and } \alpha(A^+) = \alpha(A)^+ \text{ for all } A \in \omega \text{ (see Step 1).}$$

It follows from Proposition 6.2 that the function $\alpha : \omega \to X$ is an isomorphism from the set ω onto the set X. □

6.5 Theorem. *Let* A *and* B *be two Peano sets.*

(a) The Peano sets A *and* B *are isomorphic.*

(b) There exists exactly one isomorphism $\alpha : A \to B$ *from the Peano set* A *onto the Peano set* B*.*

Proof. (a) Let ω be the minimal successor set. By Theorem 6.4, there exist two isomorphisms $\alpha : \omega \to A$ and $\beta : \omega \to B$. By Proposition 6.3, the function $\gamma := \beta \circ \alpha^{-1} : A \to B$ is an isomorphism from the Peano set A onto the Peano set B.

(b) Let $\alpha : A \to B$ and $\beta : A \to B$ be two isomorphisms from the Peano set A onto the Peano set B.

Let $Z := \{x \in A \mid \alpha(x) = \beta(x)\}$. We have to show that $Z = A$:

Since $\alpha(0_A) = 0_B$ and $\beta(0_A) = 0_B$, we have $\alpha(0_A) = \beta(0_B)$, that is, the element 0_A is contained in the set Z.

Let z be an element of the set Z. Then we have

$$\alpha(z^+) = \alpha(z)^+ = \beta(z)^+ = \beta(z^+), \text{ that is, } z^+ \in Z.$$

It follows from Axiom (P5) that $Z = A$ implying that $\alpha = \beta$. □

7 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Dedekind, Richard (1888). *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg (cit. on p. 11).

— (1932). *Gesammelte mathematische Werke*. Ed. by Robert Fricke, Emmy Noether, and Öystein Ore. Braunschweig: Vieweg. There are three volumes: Volume 1: (1930), Volume 2: (1931), Volume 3: (1932). (Cit. on pp. 11, 17).

Heath, Thomas, ed. (1908). *Euclid: The Thirteen Books of the Elements*. Vol. 2. Cambridge: Cambridge University Press. The second volume contains books III to IX. (Cit. on p. 11).

Pascal, Blaise (1665). *Traité du Triangle Arithmétique avec quelques autres petits Traités sur la même Matière*. Paris: Guillaume Desprez (cit. on p. 17).

Peano, Giuseppe (1889). *Arithmetices Prinicipia - Nova Methodo Exposita*. Romae and Florentiae: Augustae Taurinorum. This book is published under the name Ioseph Peano. (Cit. on p. 14).

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Zermelo, Ernst (1908). "Untersuchungen über die Grundlagen der Mengenlehre I". In: *Mathematische Annalen* 65, pp. 261–281 (cit. on pp. 12, 21).

— (1967). "Investigations in the Foundations of Set Theory I". In: *From Frege to Gödel. A Source Book in Mathematical Logic, 1879 - 1931*. Ed. by van Heijenoort, pp. 199–215. This article is a translation of [Zermelo 1908] into English. (Cit. on p. 12).

8 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *The Mathematical Universe*. Version 1.0.2. URL: https://www.math-garden.com/unit/nst-universe#nst1-sec-download (cit. on pp. 3, 12).

— (2020b). *Unions and Intersections of Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-unions#nst-unions-download (cit. on pp. 3, 7, 9).

— (2020c). *Functions and Equivalent Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-functions#nst-functions-download (cit. on pp. 13, 15).

— (2020d). *Families and the Axiom of Choice*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-families#nst-families-download (cit. on p. 3).

— (2020e). *The Natural Numbers and the Principle of Induction*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-natural-numbers#nst-natural-numbers-download (cit. on pp. 3–5, 9).

— (2020f). *The Binimial Coefficients and the Triangle of Pascal*. Version 1.0.0. In preparation (cit. on p. 17).

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If you are willing to share comments and ideas to improve the present unit or hints about further references, we kindly ask you to send a mail to info@math-garden.com or to use the contact form on www.math-garden.com. Contributions are highly appreciated.

Index

Symbols

A

D

F

I

M

P

R

S

T

V

Z