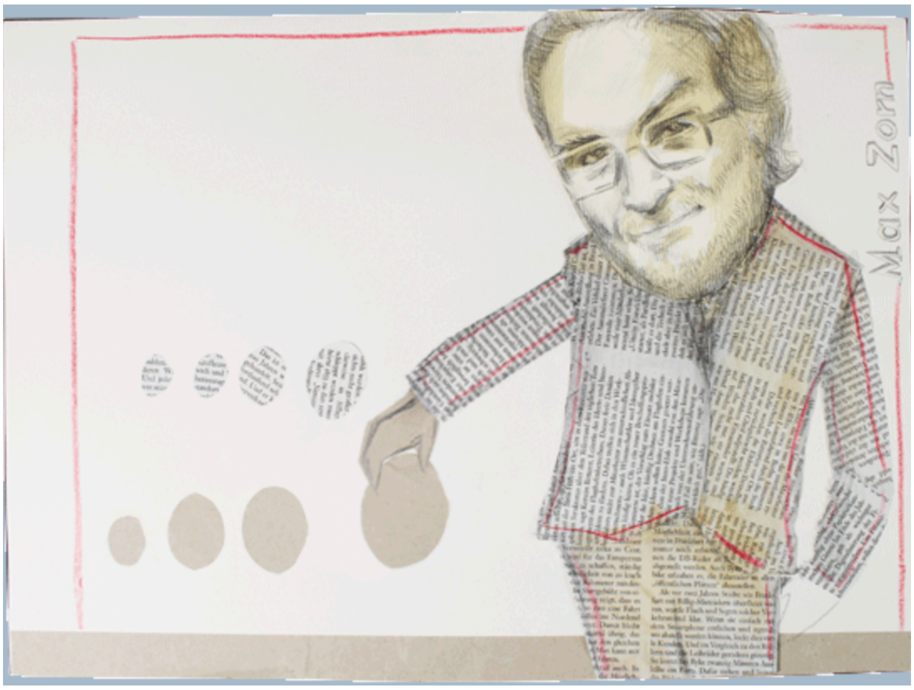
M. Garden

Ordered Sets and The Lemma of Zorn



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Contents

1 Introduction

The present unit is part of the walk *The Axioms of Zermelo and Fraenkel*. It explains the concept of ordered sets and the Lemma of Zorn.

Ordered Sets (see Section 2):

Mathematics often uses features of everyday life and transforms them into an exact and well defined mathematical context. Ordered sets are a good example for such a procedure.

The starting point are observations like *Tom is bigger than Mary*, *Karl has more money than Tom* or *Mary is happier than Karl*. The method to transform these observations into mathematics are relations: A **relation** on a set A is a subset R of the direct product $A \times A$. Two elements x and y of the set A are called **related with respect to the relation** R if we have

$$(x, y) \in R.$$

Relations are explained in detail in Unit *Direct Products and Relations* [Garden 2020c].

Order relations are relations with additional properties that are inspired by the comparison *smaller than or equal to*.

Firstly, any object is smaller than or equal to itself, hence we require

$$xRx \text{ for all } x \in A.$$

Secondly, if we have the situation that the height of one house H_1 is smaller than or equal to the height of another house H_2, and if the height of the house H_2 is smaller than or equal to the height of the house H_1, then the heights of the houses H_1 and H_2 are equal.

Hence, our second requirement for an order relation R is as follows: If x and y are two elements of the set A such that xRy and yRx, then we have $x = y$.

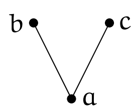
Finally, our third requirement corresponds to the following observation: If x is smaller than or equal to y and if y is smaller than or equal to z, then x is smaller than or equal to z. In other words, if xRy and yRz, then we have xRz.

In general, we use the more intuitive symbol $\leqslant$ for an order relation such that our requirements become:

$$\begin{array}{ll} (1) & x \leqslant x \text{ for all } x \in A. \\ (2) & x \leqslant y \text{ and } y \leqslant x \text{ imply } x = y \text{ for all } x, y \in A. \\ (3) & x \leqslant y \text{ and } y \leqslant z \text{ imply } x \leqslant z \text{ for all } x, y, z \in A. \end{array}$$

For more details see Definition 2.1. Note that the relation $\leqslant$ is still a subset of the direct product $A \times A$.

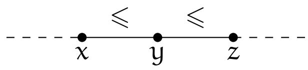
We do not require that every two elements of an ordered set may be compared. For example, the set



$$A := \{a, b, c\}$$

with $a \leqslant b$ and $a \leqslant c$ is an ordered set, but we neither have $b \leqslant c$ nor $c \leqslant b$. Therefore an order is also called a **partial order**. The set theoretical inclusion $\subseteq$ is a typical example for a partial order.

In a **totally ordered** set A we have $x \leqslant y$ or $y \leqslant x$ for all elements x and y of the set A. Hence, we can order all elements of the set A linearly.



Given a subset B of an ordered set A the set B inherits the order of the set A (see Proposition 2.3).

Often the “simple” substructures play a crucial role for the development of a mathematical theory. Such a simple substructure of an ordered set is a chain: A **chain** is a totally ordered subset of an ordered set (see Definition 2.6). Chains will play a main role in the Lemma of Zorn (see Theorem 6.3).

Given an order on a set A often the question arises whether there exist maximal elements in the set A. For example, the number 3 is the maximal element of the set $\{1, 2, 3\}$ with the natural order, and the elements b and c are the maximal elements of the set $\{a, b, c\}$ with the order $a \leqslant b$ and $a \leqslant c$ (see Definition 2.10).

The open interval $]0, 1[$ does not have a maximal element, but within the set $\mathbb{R}$ of the real numbers the set $]0, 1[$ has upper bounds like 1 or 2. The upper bound 1 is the smallest upper bound which is called a supremum. More formally, if B is a subset of a set A. then we have

$$\begin{aligned}
&m \textbf{ maximum of } B &&:\Leftrightarrow && m \in B \text{ and } x \leqslant m \text{ for all } x \in B\\
&u \textbf{ upper bound of } B &&:\Leftrightarrow && u \in A \text{ and } x \leqslant u \text{ for all } x \in B\\
&s \textbf{ supremum of } B &&:\Leftrightarrow && s \in A \text{ and } x \leqslant s \text{ for all } x \in B \text{ and}\\
&&&&& \big(x \leqslant t \text{ for all } x \in B \Rightarrow s \leqslant t\big)
\end{aligned}$$

(see Definition 2.10).

In Unit *Functions and Equivalent Sets* [Garden 2020d] we have defined extensions of functions. The main idea is that we have a family of functions $f_i : A_i \to B_i$ from some sets A_i into some sets B_i and that we want to extend these functions to *one* function

$$f : \bigcup_{i \in I} A_i \to \bigcup_{i \in I} B_i.$$

For that purpose we need a totally ordered index set I, two chains $\big(A_i\big)_{i \in I}$ and $\big(B_i\big)_{i \in I}$ with the property that

$$A_i \subseteq A_j \text{ and } B_i \subseteq B_j \text{ if } i \leqslant j$$

and functions $f_i : A_i \to B_i$ from sets A_i into sets B_i with the property that for each two elements i and j of the set I such that $i \leqslant j$ the function $f_i : A_i \to B_i$ is induced by the function $f_j : A_j \to B_j$, that is, we have

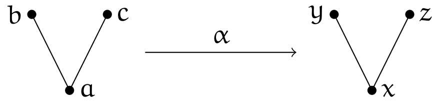
$$f_i(x) = f_j(x) \text{ for all } x \in A_i.$$

For more details see Theorem 2.18.

Isomorphisms of Ordered Sets (see Section 3):

Given two ordered sets

$$A := \{a, b, c\} \text{ with } a \leqslant b \text{ and } a \leqslant c \text{ and}$$
$$B := \{x, y, z\} \text{ with } x \leqslant y \text{ and } x \leqslant z$$



the sets are more or less identical if we look at the following correspondence:

$$\alpha : a \mapsto x,\ b \mapsto y,\ c \mapsto z.$$

The mapping $\alpha : A \to B$ is called an **isomorphism of ordered sets**, and it allows to transfer properties from an ordered set A to all ordered sets B isomorphic to the set A. See Definition 3.1 for more details.

An isomorphism from an ordered set A onto itself is called an **automorphism** of the set A. The set of the automorphisms of an ordered set A forms a subgroup of the group of all bijective mappings from the set A onto itself (see Theorem 3.10).

Initial Segments (see Section 4):

Given an ordered set A and an element a of the set A the **initial segment** A_a is defined to be the set

$$A_a := \{x \in A \mid x < a\}.$$

In other words the initial segment is a "first part" of the set A (see Definition 4.1). Initial segments will play an important role in the context of well ordered sets (see Unit *Well Ordered Sets* [**nst-well-ordered-sets**]) and in the context of ordinal numbers (see Unit *Ordinal Numbers* [**nst-ordinal-numbers**]).

In the present unit we will only introduce some elementary properties of initial segments such as:

The mapping $x \mapsto A_x$ is a bijective mapping from a totally ordered set A onto the set of the initial segments of the set A (see Proposition 4.2).

Chains of Ordered Sets (see Section 5):

A chain of ordered sets is a family $(A_i, \leqslant_i)_{i \in I}$ of ordered sets where one set A_i is embedded in the "next" set A_j. More formally, a chain of ordered sets is defined as follows:

Let I be a totally ordered index set. A family $(A_i, \leqslant_i)_{i \in I}$ is called a **chain of ordered sets** if the following conditions are fulfilled:

(i) The family $(A_i)_{i \in I}$ is a chain, that is, we have $A_i \subseteq A_j$ whenever $i \leqslant j$.

(ii) For each two elements i and j of the set I such that $i \leqslant j$, the order $\leqslant_i$ on the set A_i is induced by the order $\leqslant_j$ on the set A_j, that is, we have

$$x \leqslant_i y \text{ if and only if } x \leqslant_j y \text{ for all } x, y \in A_i.$$

See Definition 5.1. An important property of chains of ordered sets is that we can move from the family $(A_i, \leqslant_i)_{i \in I}$ of ordered sets to the set

$$A := \bigcup_{i \in I} A_i$$

with an order $\leqslant$ on the set A in such a way that the set $A = (A, \leqslant)$ is an ordered set and that the orders $\leqslant_i$ of the sets A_i are all induced by the order $\leqslant$. In other words, the orders $\leqslant_i$ can be replaced by one order $\leqslant$. See Theorem 5.2.

A similar construction deals with initial segments: If the set A_i is an initial segment of the set A_j whenever $i < j$, then we can simplify this setting by the observation that each set A_i is an initial segment of the set $A := \bigcup_{i \in I} A_i$ (or $A = A_i$). For more details see Theorem 5.6.

Finally, under suitable conditions, one can extend isomorphisms $\alpha_i : A_i \to B_i$ where $\left(A_i\right)_{i \in I}$ and $\left(B_i\right)_{i \in I}$ are chains of ordered sets to one isomorphism

$$\alpha : A := \bigcup_{i \in I} A_i \to B := \bigcup_{i \in I} B_i.$$

Fore more details see Theorem 5.7. All these results will be helpful in the investigation of well ordered sets and ordinal numbers.

The Lemma of Zorn (see Section 6):

The Lemma of Zorn is by far the most important result of the present unit. It is *the* tool for many existence theorems in mathematics such as

- the existence of a well ordering on an arbitrary set (see Unit *Well Ordered Sets* [**nst-well-ordered-sets**]),
- the existence of a base in an arbitrary vector space (see Unit *Vector Spaces* [Garden 2021a]) or
- the existence of a maximal ideal in an arbitrary unitary ring (see Unit *Ideals in Rings* [Garden 2021b]).

The Lemma of Zorn guarantees the existence of maximal elements in an ordered set A provided that every chain of the set A has an upper bound in the set A. More precisely:

Let $A = (A, \leqslant)$ be an ordered set such that every chain C of the set A has an upper bound in the set A. Then the set A contains a maximal element (see Theorem 6.3).

The Lemma of Zorn is a consequence of the axiom of choice explained in Unit *Families and the Axiom of Choice* [Garden 2020e]. In fact, both assertions are equivalent.

2 Ordered Sets

Definition of Partial and Total Orders:

Ordered sets are sets with a specific sort of relations. Recall that a relation R on a set A is a subset of the direct product $A \times A$. Fore more details about relations and direct products see Unit *Direct Products and Relations* [Garden 2020c].

2.1 Definition. Let A be a set, and let R be a relation on the set A.

(a) The relation R is called an **order** or, equivalently, a **partial order on the set** A if it fulfills the following conditions:

(i) The relation R is **reflexive**, that is, xRx for all elements x of the set A.

(ii) The relation R is **antisymmetric**, that is, xRy and yRx imply $x = y$ for all elements x and y of the set A.

(iii) The relation R is **transitive**, that is, xRy and yRz imply xRz for all elements x, y and z of the set A.

(b) A partial order is often denoted by $\subseteq$ or by $\leqslant$.

(c) A set A with a (partial) order R or $\leqslant$ is called an **ordered set** or, equivalently, a **partially ordered set** and is often denoted by $\mathcal{A} = (A, R)$ or by $\mathcal{A} = (A, \leqslant)$.

(d) A (partial) order $\leqslant$ on a set A is called a **total order** if we have $x \leqslant y$ or $y \leqslant x$ for all elements x and y of the set A. A set A with a total order $\leqslant$ is called a **totally ordered set**.

French / German. (Partially) Ordered Set = Ensemble (partiellement) ordonné = (Partiell) geordnete Menge. Totally ordered set = Ensemble totalement ordonné = Vollständig geordnete Menge.

2.2 Remark. A totally ordered set can be visualized as follows:

$\leqslant$

Induced Orders:

The following proposition explains how a subset B of an ordered set A inherits the order of the set A.

2.3 Proposition. *Let* $\mathcal{A} = (A, \leqslant_A)$ *be an ordered set, and let* B *be a subset of the set* A.

Then there exists exactly one order $\leqslant_B$ *on the set* B *such that*

$$x \leqslant_B y \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in B.$$

Proof. The relation $\leqslant_A$ is a subset R of the direct product $A \times A$. Let

$$S := R \cap (B \times B) = (R \cap B) \times (R \cap B).^{1}$$

If we denote the relation S on the set B by $\leqslant_B$, then we have

$$x \leqslant_B y \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in B. \tag{1}$$

Conversely, if $\leqslant_B$ is an order on the sets B fulfilling Condition (1), then we have

$$\leqslant_B = R \cap (B \times B) = S.$$

□

[1]This equation is an elementary property of direct products. Fore more details see Unit *Direct Products and Relations* [Garden 2020c].

2.4 Definition. Let $A = (A, \leqslant_A)$ be an ordered set, let B be a subset of the set A, and let $\leqslant_B$ be the order on the set B such that

$$x \leqslant_B y \text{ if and only if } x \leqslant_A y \text{ for all } x, y \in B \text{ (Proposition 2.3)}.$$

The order $\leqslant_B$ is called **the order on the set B induced by the order $\leqslant_A$**.

Often we write $A = (A, \leqslant)$ and $B = (B, \leqslant)$ without explicitly distinguishing between the order $\leqslant_A$ on the set A and the induced order $\leqslant_B$ on the set B.

French / German. Induced order = Ordre induit = Induzierte Ordnung.

2.5 Proposition. *Let $A = (A, \leqslant)$ be an ordered set, and let $B = (B, \leqslant)$ be a subset of the set A with the induced order.*

If the pair $A = (A, \leqslant)$ is totally ordered, then the pair $B = (B, \leqslant)$ is also totally ordered.

Proof. The proof is obvious. □

2.6 Definition. A subset C of an ordered set $A = (A, \leqslant)$ is called a **chain** if the subset $C = (C, \leqslant)$ is totally ordered with respect to the order induced by the order of the set A. In particular, a totally ordered set is a chain.

French / German. Chain = Chaîne = Kette.

Elementary Properties of Ordered Sets:

2.7 Proposition. *The pair $(\emptyset, R)$ is an ordered set if and only if $R = \emptyset$. The pair $(\emptyset, \emptyset)$ is even a totally ordered set.*

Proof. If the pair $(\emptyset, R)$ is an ordered set, then we have $R \subseteq \emptyset \times \emptyset = \emptyset$[2].
Obviously, the pair $(\emptyset, \emptyset)$ is a totally ordered set. □

2.8 Definition. Let $A = (A, \leqslant)$ be an ordered set, and let a and b be two elements of the set A.

(a) Set $a < b$ if $a \leqslant b$ and $a \neq b$.

(b) Set $a \geqslant b$ if $b \leqslant a$.

(c) Set $a > b$ if $a \geqslant b$ and $a \neq b$, that is, if $b < a$.

2.9 Proposition. *Let $A = (A, \leqslant)$ be an ordered set.*

(a) The relation $\geqslant$ defined in Definition 2.8 is an order on the set A.

(b) If the relation $\leqslant$ is a total order, then the relation $\geqslant$ is also a total order.

(c) If a, b and c are three elements of the set A, then we have

(i) If $(a \leqslant b)$ and $(b < c)$, then we have $(a < c)$.

(ii) If $(a < b)$ and $(b \leqslant c)$, then we have $(a < c)$.

[2] For a proof of the equation $\emptyset \times \emptyset = \emptyset$ see Unit *Direct Products and Relations* [Garden 2020c].

(iii) If $(a \geqslant b)$ *and* $(b > c)$*, then we have* $(a > c)$*.*

(iv) If $(a > b)$ *and* $(b \geqslant c)$*, then we have* $(a > c)$*.*

Proof. Let a, b and c be three elements of the set A.

(a) (i) It follows from $a \leqslant a$ that $a \geqslant a$.

(ii) If $a \geqslant b$ and $b \geqslant a$, then we have $b \leqslant a$ and $a \leqslant b$ implying that $a = b$.

(iii) If $a \geqslant b$ and $b \geqslant c$, then we have $c \leqslant b$ and $b \leqslant a$ implying that $c \leqslant a$, that is, $a \geqslant c$.

(b) Let a and b be two elements of the set A. Since the relation $\leqslant$ is a total order, we have $a \leqslant b$ or $b \leqslant a$ implying that $b \geqslant a$ or $a \geqslant b$. Hence, the relation $\geqslant$ is a total order on the set A.

(c) (i) It follows from $b < c$ that $b \leqslant c$. Since $a \leqslant b$ and $b \leqslant c$, we have $a \leqslant c$. Assume that $a = c$. It follows that $c \leqslant a$. It follows from $c \leqslant a$ and $a \leqslant b$ that $c \leqslant b$. Since $c \leqslant b$ and $b \leqslant c$, we get $b = c$, in contradiction to $b < c$.

(ii) - (iv) follow as in (i). □

Maxima and Suprema of Ordered Sets:

2.10 Definition. Let $A = (A, \leqslant)$ be an ordered set, and let B be a subset of the set A.

(a) An element b of the set B is called a **maximal element** or a **maximum** of the set B if we have

$$x \leqslant b \text{ for all } x \in B.$$

Note that the element b (if it exists) is contained in the set B.

(b) An element b of the set B is called a **minimal element** or a **minimum** of the set B if we have

$$x \geqslant b \text{ for all } x \in B.$$

Note that the element b (if it exists) is contained in the set B.

(c) An element a of the set A is called an **upper bound** of the set B if we have

$$x \leqslant a \text{ for all } x \in B.$$

Note that the element a (if it exists) is contained in the set A, but not necessarily in the set B.

(d) An element a of the set A is called a **lower bound** of the set B if we have

$$x \geqslant a \text{ for all } x \in B.$$

Note that the element a (if it exists) is contained in the set A, but not necessarily in the set B.

(e) An element a of the set A is called a **supremum** of the set B if the following conditions are fulfilled:

(i) The element a is an upper bound of the set B.

(ii) If c is a second upper bound of the set B, then we have $a \leqslant c$. In other words, the element a is the smallest upper bound of the set B.

Note that that the supremum a (if it exists) is an element of the set A, but not necessarily of the set B.

(f) An element a of the set A is called an **infimum** of the set B if the following conditions are fulfilled:

(i) The element a is a lower bound of the set B.

(ii) If c is a second lower bound of the set B, then we have $a \geqslant c$. In other words, the element a is the biggest lower bound of the set B.

Note that that the infimum a (if it exists) is an element of the set A, but not necessarily of the set B.

French / German. Maximum = Maximum = Maximum. Minimum = Minimum = Minimum. Maximal element = Élément maximal = Maximales Element. Minimal element = Élément minimal = Minimales Element. Upper bound = Majorant = Obere Schranke. Lower bound = Minorant = Untere Schranke. Supremum = Borne supérieure (or supremum) = Supremum. Infimum = Borne inférieure (or infimum) = Infimum.

2.11 Example. Let A be a set, let $\mathcal{P}(A)$ be the power set of the set A, and let $\subseteq$ denote the subset relation. Then the pair $\big(\mathcal{P}(A), \subseteq\big)$ is a partially ordered set.

The set A is the only maximal element of the set $\mathcal{P}(A)$, and the empty set $\emptyset$ is the only minimal element of the set $\mathcal{P}(A)$.

2.12 Proposition. *Let $A = (A, \leqslant)$ be an ordered set, and let B be a subset of the set A.*

(a) If the set B admits a maximum, then the maximum is unique.

(b) If the set B admits a minimum, then the minimum is unique.

(c) If the set B admits a supremum in A, then the supremum is unique.

(d) If the set B admits an infimum in A, then the infimum is unique.

Proof. (a) Let r and s be two maximal elements of the set B. Then we have $r \leqslant s$ and $s \leqslant r$ implying that $r = s$.

(b) The assertion follows as in (a).

(c) Let r and s be two suprema of the set B. Since we have $x \leqslant r$ and $x \leqslant s$ for all elements x of the set B and since the elements r and s of the set A are suprema of the set B, it follows that $r \leqslant s$ and $s \leqslant r$ implying that $r = s$.

(d) The assertion follows as in (c). $\square$

Direct Products of Chains:

2.13 Proposition. *Let I be a totally ordered index set, and let $(A_i)_{i \in I}$ be a family of sets such that the set A_j is a subset of the set A_k if $j \leqslant k$. Then we have*

$$\Big(\bigcup_{i \in I} A_i\Big) \times \Big(\bigcup_{i \in I} A_i\Big) = \bigcup_{i \in I} (A_i \times A_i).$$

Proof. Step 1. *We have $\big(\bigcup_{i \in I} A_i\big) \times \big(\bigcup_{i \in I} A_i\big) \subseteq \bigcup_{i \in I} (A_i \times A_i)$:*

For, let (x, y) be an element of the set $\left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{i \in I} A_i\right)$. Then there exist two elements j and k of the set I such that the pair (x, y) is contained in the direct product $A_j \times A_k$. Since the set I is totally ordered, we may suppose w.l.o.g. that $j \leqslant k$. It follows that the set A_j is a subset of the set A_k implying that

$$(x, y) \in A_k \times A_k \subseteq \bigcup_{i \in I} (A_i \times A_i).$$

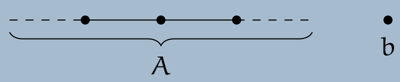
Step 2. *We have* $\bigcup_{i \in I} (A_i \times A_i) \subseteq \left(\bigcup_{i \in I} A_i\right) \times \left(\bigcup_{i \in I} A_i\right)$*:*
The assertion is obvious. □

One Point Extensions:

One point extensions will be needed in the study of ordinal sets. See Unit *Ordinal Numbers* [**nst-ordinal-numbers**].

2.14 Definition. Let $A = (A, \leqslant_A)$ be an ordered set, and let b be an element not contained in the set A. Set $B := A \mathbin{\dot\cup} \{b\}$. We define an order $\leqslant_B$ on the set B as follows:

For each two elements x and y of the set A, set $x \leqslant_B y$ if and only if $x \leqslant_A y$. For each element x of the set B, set $x \leqslant_B b$. The pair $(B, \leqslant_B)$ is called the **one point extension** of the pair $(A, \leqslant_A)$.



French / German. The notion *one point extension* is probably no standard notion in mathematics. It is based on the notion *one-point compactification* of Alexandroff. I am not aware of a corresponding French expression. One point extension = Einpunkt-Erweiterung.

2.15 Proposition. *Let $(A, \leqslant_A)$ be an ordered set, and let $(B, \leqslant_B)$ be a one point extension of the pair $(A, \leqslant)$ where b is an element of the set B not contained in the set A.*

(a) The order $\leqslant_A$ of the set A is induced by the order $\leqslant_B$ of the set B.

(b) The pair $(A, \leqslant_A)$ is totally ordered if and only if the pair $(B, \leqslant_B)$ is totally ordered.

Proof. Both assertions follow immediately from Definition 2.14. □

Extensions of Functions:

We recall the definition of a function. For more details see Unit *Functions and Equivalent Sets* [Garden 2020d].

2.16 Definition. Let A and B be two sets.

(a) A **function** $f : A \to B$ **from the set** A **into the set** B is a triple (f, A, B) where the set f is a subset of the direct product $A \times B$ with the following property:

For each element x of the set A there is exactly one element y of the set B such that the pair (x, y) is contained in the set f.

(b) Let $f : A \to B$ be a function from the set A into the set B, and let x be an element of the set A. The unique element y of the set B such that the pair (x, y) is contained in the set f is denoted by $y = f(x)$. We also write $f : x \mapsto y$ or, equivalently, $f : x \mapsto f(x)$.

2.17 Theorem. *Let A and B be two sets, and let $\alpha : A \to B$ and $\beta : A \to B$ be two functions from the set A into the set B. Then we have*

$$\alpha = \beta \text{ if and only if } \alpha(x) = \beta(x) \text{ for all } x \in A.$$

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020d]. □

2.18 Theorem. *Let I be a totally ordered index set, and let $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ be two families of sets with the following properties:*

(i) The families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are chains, that is, we have

$$A_i \subseteq A_j \text{ and } B_i \subseteq B_j \text{ if } i \leqslant j.$$

(ii) For each element i of the set I, there exists a function $\alpha_i : A_i \to B_i$ from the set A_i into the set B_i.

(iii) For each two elements i and j of the set I such that $i \leqslant j$ the function $\alpha_i : A_i \to B_i$ is induced by the function $\alpha_j : A_j \to B_j$, that is, we have

$$\alpha_i(x) = \alpha_j(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i\in I} A_i \text{ and } B := \bigcup_{i\in I} B_i.$$

(a) There exists exactly one function $\alpha : A \to B$ from the set $A = \bigcup_{i\in I} A_i$ into the set $B = \bigcup_{i\in I} B_i$ such that $\alpha|_{A_i} = \alpha_i$ for all elements i of the set I.

(b) If the functions $\alpha_i : A_i \to B_i$ are injective for all elements i of the set I, then the function $f : A \to B$ is also injective.

(c) If the functions $\alpha_i : A_i \to B_i$ are surjective for all elements i of the set I, then the function $f : A \to B$ is also surjective.

(d) If the functions $\alpha_i : A_i \to B_i$ are bijective for all elements i of the set I, then the function $f : A \to B$ is also bijective.

Proof. Note that the unions $A := \bigcup_{i\in I} A_i$ and $B := \bigcup_{i\in I} B_i$ exist (for more details see Unit *Families and the Axiom of Choice* [Garden 2020e]).

(a) Step 1. *Definition of the function $\alpha : A \to B$:*

Note that for each element i of the set I, the function $\alpha_i : A_i \to B_i$ is a subset of the set $A_i \times B_i$. Set

$$\alpha := \bigcup_{i\in I} \alpha_i.$$

By Proposition 2.13, we have

$$\alpha = \bigcup_{i\in I} \alpha_i \subseteq \bigcup_{i\in I} (A_i \times B_i)) = \Big(\bigcup_{i\in I} A_i\Big) \times \Big(\bigcup_{j\in I} B_j\Big) = A \times B.$$

Let x be an element of the set $A = \bigcup_{i \in I} A_i$. Then there exists an element i_x of the set I such that the element x is contained in the set A_{i_x}. It follows that

$$(x, \alpha_{i_x}(x)) \in \alpha_{i_x} \subseteq \bigcup_{i \in I} \alpha_i = \alpha.$$

If y is a further element of the set B such that the pair (x, y) is contained in the set α, then there exists an element j_x of the set I such that the pair (x, y) is contained in the set α_{j_x}, that is, $\alpha_{j_x}(x) = y$.

Since the set I is totally ordered, we have $i_x \leqslant j_x$ or $j_x \leqslant i_x$. In both cases it follows that

$$y = \alpha_{j_x}(x) = \alpha_{i_x}(x)$$

implying that the element $\alpha_{i_x}(x)$ is the only element y of the set B such that the pair (x, y) is contained in the set α. In other words, the set α is a function $\alpha : A \longrightarrow B$.

Step 2. *We have* $\alpha|_{A_i} = \alpha_i$ *for all elements* i *of the set* I*:*

The assertion follows from the definition of the function $\alpha : A \longrightarrow B$.

Step 3. *The function* $\alpha : A \longrightarrow B$ *is uniquely determined:*

Let $\beta : A \longrightarrow B$ be a second function from the set A into the set B such that $\alpha_i = \beta|_{A_i}$ for all elements i of the set I. Let x be an element of the set $A = \bigcup_{i \in I} A_i$. Then there exists an element $i = i_x$ of the set I such that the element x is contained in the set A_i. It follows that

$$\beta(x) = \alpha_i(x) = \alpha(x)$$

implying that $\beta = \alpha$ (Theorem 2.17).

(b) We have to show that the function $\alpha : A \longrightarrow B$ is injective:

For, let x and y be two elements of the set $A = \bigcup_{i \in I} A_i$ such that $\alpha(x) = \alpha(y)$. It follows from Assumption (i) that there exists an element j of the set I such that the elements x and y are both contained in the set A_j. By Step (a), we have

$$\alpha_j(x) = \alpha(x) = \alpha(y) = \alpha_j(y).$$

Since the function $\alpha_j : A_j \longrightarrow B_j$ is injective, we get $x = y$.

(c) We have to show that the function $\alpha : A \longrightarrow B$ is surjective:

For, let z be an element of the set $B = \bigcup_{i \in I} B_i$. Then there exists an element j of the set I such that the element z is contained in the set B_j. Since the function $\alpha_j : A_j \longrightarrow B_j$ is surjective, there exists an element x of the set A_j such that $z = \alpha_j(x)$. It follows from (a) that $\alpha(x) = \alpha_j(x) = z$.

(d) follows from (b) and (c). □

2.19 Remark. Theorem 2.18 allows to replace the functions $\alpha_i : A_i \longrightarrow B_i$ by *one* function $\alpha : A \longrightarrow B$.

Historical Notes:

Partially ordered sets have been introduced by Felix Hausdorff:

Nehmen wir an, zwischen je zwei verschiedenen Elementen a, b *einer Menge* A *bestehe jetzt nicht mehr, wie bei geordneten Mengen, eine und nur eine von zwei Beziehungen* ($a < b$, $a > b$)*, sondern eine und nur eine von drei Beziehungen*

$$a < b, \ a > b, \ a \parallel b,$$

die wir lesen wollen: a *vor* b, a *nach* b, a *unvergleichbar mit* b. *Von den beiden ersten setzen wir dieselben Eigenschaften wie im Falle geordneter Mengen voraus, was für die dritte Beziehung notwendig ihre Symmetrie zur Folge hat, d. h.*

$$\textit{aus } a < b, \ a > b, \ a \parallel b \textit{ folgt } b > a, \ b < a, \ b \parallel a;$$

$$\textit{aus } a < b, \ b < c \textit{ folgt } a < c \ \textit{(transitives Gesetz)}.$$

Eine solche Menge heißt eine **teilweise geordnete Menge***; die geordneten Mengen sind Spezialfälle der teilweise geordneten, nämlich wenn Paare unvergleichbarer Elemente nicht existieren [...]*

See [Hausdorff 1914, p. 139].

Suppose that between two different elements a, b *of a set* A *no longer exist, as with ordered sets, one and only one of two relationships* ($a < b$, $a > b$)*, but one and only one of three relationships*

$$a < b, \ a > b, \ a \parallel b,$$

we want to read: a *before* b, a *after* b, a *incomparable with* b. *From the first two we assume the same properties as in the case of ordered sets, which necessarily results in their symmetry for the third relationship, i.e.*

$$\textit{from } a < b, \ a > b, \ a \parallel b \textit{ follows } b > a, \ b < a, \ b \parallel a;$$

$$\textit{from } a < b, \ b < c \textit{ follows } a < c \ \textit{(transitive law)}.$$

Such a set is called a **partially ordered set***; the ordered sets are special cases of the partially ordered ones, namely when pairs of incomparable elements do not exist [...]*

(Translation by the author.)

3 Isomorphisms of Ordered Sets

3.1 Definition. Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \to B$ be a function from the set A into the set B.

(a) The function $\alpha : A \to B$ is called a **homomorphism of ordered sets** if we have

$$x \leqslant_A y \text{ implies } \alpha(x) \leqslant_B \alpha(y) \text{ for all } x, y \in A.$$

A homomorphism of ordered sets is also called a **homomorphism** if no danger of confusion arises.

(b) An injective homomorphism $\alpha : A \to B$ is called a **monomorphism of ordered sets**.

(c) A surjective homomorphism $\alpha : A \to B$ is called an **epimorphism of ordered sets**.

(d) A bijective homomorphism $\alpha : A \to B$ with the property that the inverse function $\alpha^{-1} : B \to A$ is also a homomorphism is called an **isomorphism of ordered sets**.

(e) The ordered sets $(A, \leqslant_A)$ and $(B, \leqslant_B)$ are called **isomorphic** if there exists an isomorphism $\alpha : A \to B$ from the set A onto the set B. If the sets A and B are isomorphic, then we write $A \cong B$.

(f) An isomorphism $\alpha : A \to A$ from the set A onto itself is called an **automorphism of the set** A.

French / German. Homomorphism = Homomorphisme (or morphisme) = Homomorphismus. Monomorphism = Monomorphisme = Monomorphismus. Epimorphism = Épimorphisme = Epimorphismus. Isomorphism = Isomorphisme = Isomorphismus. Automorphism = Automorphisme = Automorphismus.

3.2 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two ordered sets, and let* $\alpha : A \to B$ *be a mapping from the set* A *into the set* B. *Then the mapping* $\alpha : A \to B$ *is an isomorphism of the set* A *onto the set* B *if and only if the mapping* $\alpha : A \to B$ *is bijective and if we have*

$$\alpha(x) \leqslant_B \alpha(y) \textit{ if and only if } x \leqslant_A y \textit{ for all } x, y \in A.$$

Proof. The proof is obvious. □

3.3 Example. Let $A = (A, \leqslant)$ be the ordered set $A := \{a, b, c\}$ such that $a \leqslant b$ and $a \leqslant c$. Then the mapping $\alpha : A \to A$ defined by $\alpha : a \mapsto a$, $b \mapsto c$ and $c \mapsto b$ is an automorphism of the ordered set A.

Elementary Properties of Isomorphisms between Ordered Sets:

3.4 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two ordered sets, and let* $\alpha : A \to B$ *be an isomorphism. Then we have*

$$x <_A y \textit{ if and only if } \alpha(x) <_B \alpha(y) \textit{ for all } x, y \in A.$$

Proof. The proof is obvious. □

3.5 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two ordered sets, and let* $\alpha : A \to B$ *be an isomorphism from the set* A *onto the set* B.

Then the function $\alpha^{-1} : B \to A$ *is an isomorphism from the set* B *onto the set* A.

Proof. Obviously, the function $\beta : B \to A$ is bijective. Let r and s be two elements of the set B. Then there exist two elements x and y of the set A such that $\alpha(x) = r$ and $\alpha(y) = s$. It follows that $x = \beta(r)$ and $y = \beta(s)$. Hence, we have

$$r \leqslant_B s \text{ if and only if } \alpha(x) \leqslant_B \alpha(y) \text{ if and only if } x \leqslant_A y \text{ if and only if } \beta(r) \leqslant_A \beta(s).$$

□

For the proof of Proposition 3.7 we will need the following elementary property of bijective functions:

3.6 Proposition. *Let A, B and C be three sets, and let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be two functions from the set A into the set B and from the set B into the set C, respectively.*

(a) If the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are injective, then the function $g \circ f : A \longrightarrow C$ is also injective.

(b) If the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are surjective, then the function $g \circ f : A \longrightarrow C$ is also surjective.

(c) If the functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$ are bijective, then the function $g \circ f : A \longrightarrow C$ is also bijective.

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020d]. □

3.7 Proposition. *Let $(A, \leqslant_A)$, $(B, \leqslant_B)$ and $(C, \leqslant_C)$ be three ordered sets.*

(a) Suppose that there exist two isomorphisms $\alpha : A \longrightarrow B$ and $\beta : B \longrightarrow C$ from the set A onto the set B and from the set B onto the set C, respectively.

Then the composite $\gamma := \beta \circ \alpha : A \longrightarrow C$ is an isomorphism from the set A onto the set C.

(b) If $A \cong B$ and $B \cong C$, then we have $A \cong C$.

Proof. (a) By Proposition 3.6, the function $\gamma : A \longrightarrow C$ is bijective. For two elements x and y of the set A, we have

$$x \leqslant_A y \Leftrightarrow \alpha(x) \leqslant_B \alpha(y) \Leftrightarrow \beta(\alpha(x)) \leqslant_C \beta(\alpha(y)) \Leftrightarrow \gamma(x) \leqslant_C \gamma(y).$$

By Proposition 3.2, the mapping $\gamma : A \longrightarrow C$ is an isomorphism.

(b) follows from (a). □

We will need the following elementary properties of groups in Theorem 3.11:

3.8 Definition. (a) A pair $(G, *)$ consisting of a non-empty set G and an operation $* : G \times G \longrightarrow G$ on the set G is called a **group** if the following conditions are fulfilled:

(i) We have

$$x * y \in G \text{ for all } x, y \in G \text{ (closure)}.$$

(ii) We have

$$(x * y) * z = x * (y * z) \text{ for all } x, y, z \in G \text{ (associativity)}.$$

(iii) There exists an element id of the group G such that

$$x * \mathrm{id} = \mathrm{id} * x = x \text{ for all } x \in G \text{ (existence of an identity element)}.$$

(iv) For each element x of the group G there exists an element $y = y_x$, denoted by x^{-1}, of the group G such that

$$x * y = \mathrm{id} = y * x \text{ (existence of an inverse element)}.$$

A group $G = (G, *)$ is called a **commutative group** or, equivalently, an **abelian group** if we have

$$x * y = y * x \text{ for all } x, y \in G.$$

(b) If U is a non-empty subset of a group $G = (G, *)$ such that the pair $(U, *_U)$ (where $*_U$ denotes the operation on the set G restricted to the set U) is also a group, then the pair $(U, *_U)$ is called a **subgroup** of the group G. It is denoted by $U = (U, *)$.

3.9 Proposition. *Let $G = (G, *)$ be a group, and let U be a non-empty subset of the set G.*

(a) The pair $U = (U, *)$ is a subgroup of the group G if and only if the following conditions are fulfilled:

(i) We have

$$u * v \in U \text{ for all } u, v \in U.$$

(ii) For each element u of the set U the inverse u^{-1} is an element of the set U.

(b) If the group G is abelian, then the subgroup U is also abelian.

Proof. For the proof and more details about groups and subgroups see Unit *Groups and Subgroups* [Garden 2020g]. □

We recall that the set of the bijective functions from a set A onto itself forms a group:

3.10 Theorem. *Let A be a non-empty set, and let*

$$\mathcal{B}(A) := \{f : A \to A \mid f \text{ is bijective}\}$$

be the set of the bijective functions from the set A into itself.

Then the pair $(\mathcal{B}(A), \circ)$ is a group where $\circ$ denotes the composition of two functions of the set $\mathcal{B}(A)$. In general, this group is not abelian.

Proof. See Unit *Functions and Equivalent Sets* [Garden 2020d]. □

3.11 Theorem. *Let $(A, \leqslant)$ be an ordered set, and let*

$$\mathcal{B}(A) := \{f : A \to A \mid f \text{ is bijective}\}.$$

Then the set $\mathrm{Aut}(A)$ of the automorphisms of the set A is a subgroup of the group $\mathcal{B}(A)$.

Proof. By Theorem 3.10, the pair $((\mathcal{B}(A), \circ)$ is a group. We have to verify Conditions (i) and (ii) of Proposition 3.9:

(i) If $\alpha : A \to A$ and $\beta : A \to A$ are two isomorphisms, then it follows from Proposition 3.7 that the function $\beta \circ \alpha : A \to A$ is an isomorphism.

(ii) If $\alpha : A \to A$ is an isomorphism, then it follows from Proposition 3.5 that the inverse function $\alpha^{-1} : A \to A$ is an isomorphism. □

3.12 Definition. Let $(A, \leqslant)$ be an ordered set, and let G be the group of the automorphisms of the set A (Theorem 3.11).

(a) The group G is called the **full automorphism group of the ordered set** $(A, \leqslant)$. It is denoted by $\mathrm{Aut}(A)$.

(b) Every subgroup of the group G is called an **automorphism group of the ordered set** $(A, \leqslant)$.

French / German. Automorphism group of an ordered set = Groupe d'automorphismes d'un ensemble ordonné = Automorphismengruppe einer geordneten Menge.

3.13 Proposition. *Let* $(A, \leqslant)$ *be an ordered set, and let* $\mathrm{id} : A \to A$ *be the identity, that is,* $\mathrm{id}(x) = x$ *for all elements* x *of the set* A.

Then the function $\mathrm{id} : A \to A$ *is an automorphism of the set* A.

Proof. The proof is obvious. Note that the assertion also follows from Theorem 3.11. □

3.14 Proposition. *Let* $(A, \leqslant_A)$ *and* $(B, \leqslant_B)$ *be two ordered sets, and let* $\alpha : A \to B$ *be an isomorphism from the set* A *onto the set* B.

If the pair $(A, \leqslant_A)$ *is totally ordered, then the pair* $(B, \leqslant_B)$ *is totally ordered, too.*

Proof. The proof is obvious. □

Historical Notes:

Isomorphisms of ordered sets have been introduced by Georg Cantor:

Zwei geordnete Mengen M *und* N *nennen wir* **ähnlich**, *wenn sie sich gegenseitig eindeutig einander so zuordnen lassen, dass wenn* m_1 *und* m_2 *irgend zwei Elemente von* M, n_1 *und* n_2 *die entsprechenden Elemente von* N *sind, alsdann immer die Rangbeziehung von* m_1 *zu* m_2 *innerhalb* M *dieselbe ist wie die von* n_1 *und* n_2 *innerhalb* N. *Eine solche Zuordnung ähnlicher Mengen nennen wir eine* **Abbildung** *derselben aufeinander.*

See [Cantor 1895, p. 497].

We call two ordered sets M *and* N **similar** *if they can be clearly assigned to one another such that if* m_1 *and* m_2 *are any two elements of* M *and if* n_1 *and* n_2 *are the corresponding elements of* N, *then the rank relationship of* m_1 *and* m_2 *within* M *is always the same as that of* n_1 *and* n_2 *within* N. *We call such an assignment of similar sets a* **mapping** *from one set on the other.*

(Translation by the author.)

4 Initial Segments

Initial segments will play an important role in the study of ordinal numbers (see Unit *Ordinal Numbers* [**nst-ordinal-numbers**]). Here we only present some elementary properties.

Definition of Initial Segments:

4.1 Definition. Let $(A, \leqslant)$ be an ordered set, and let a be an element of the set A. Then the set

$$A_a := \{x \in A \mid x < a\}$$

is called the **initial segment of the set A with respect to the element a**.

French / German. Initial segment = Section commençante (or Segment initial) = Anfangsstück.

Elementary Properties of Initial Segments:

4.2 Proposition. *Let $(A, \leqslant)$ be an ordered set, let $\mathcal{A}$ be the set of the initial segments of the set A, and let $\alpha : A \to \mathcal{A}$ be the mapping from the set A into the set $\mathcal{A}$ defined by*

$$\alpha : x \mapsto A_x := \{z \in A \mid z < x\}.$$

If the set A is totally ordered, then the mapping $\alpha : A \to \mathcal{A}$ is bijective.

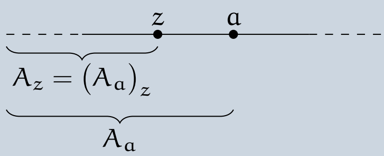
Proof. Obviously, the mapping $\alpha : A \to \mathcal{A}$ is surjective.

In order to show that the mapping $\alpha : A \to \mathcal{A}$ is injective, let x and y be two elements of the set A. Suppose that $A_x = A_y$ and assume that $x \neq y$. Since the set A is totally ordered, we have $x < y$ or $y < x$. W.l.o.g. suppose that $x < y$. It follows that the element x is contained in the set $A_y = \{z \in A \mid z < y\}$, but not in the set $A_x = \{z \in A \mid z < x\}$, in contradiction to the assumption that $A_x = A_y$. $\square$

4.3 Proposition. *Let A be an ordered set, let a be an element of the set A, and let $A_a := \{x \in A \mid x < a\}$ be the initial segment of the set A with respect to the element a. Let z be an element of the initial segment A_a.*

Then we have

$$\left(A_a\right)_z = A_z$$



where the sets A_z and $\left(A_a\right)_z$ denote the initial segments of the sets A and A_a with respect to the element z, respectively.

Proof. Since the element z is contained in the set A_a, we have $z < a$. It follows that

$$\begin{aligned} \left(A_a\right)_z &= \{x \in A_a \mid x < z\} = \{x \in A \mid x < a \text{ and } x < z\} \\ &= \{x \in A \mid x < z\} = A_z. \end{aligned}$$

$\square$

4.4 Proposition. *Let $(A, \leqslant_A)$ and $(B, \leqslant_B)$ be two ordered sets, and let $\alpha : A \to B$ be an isomorphism from the ordered set A onto the ordered set B.*

Let a be an element of the set A, and let $A_a := \{x \in A \mid x < a\}$ be the initial segment of the set A with respect to the element a. Then the set $\alpha(A_a)$ is the initial segment of the *set* B *with respect to the element* $b := \alpha(a)$*, that is,*

$$\alpha(A_a) = B_b = B_{\alpha(a)}.$$

Proof. Step 1. *We have* $\alpha(A_a) \subseteq B_b$*:*

For, let x be an element of the set A_a. Then we have $x <_A a$, and it follows from Proposition 3.4 that

$$\alpha(x) <_B \alpha(a) = b$$

implying that the element $\alpha(x)$ is contained in the set B_b.

Step 2. *We have* $B_b \subseteq \alpha(A_a)$*:*

For, let y be an element of the set B_b. Then we have $y <_b b$. Since the function $\alpha : A \to B$ is an isomorphism, there exists an element x of the set A such that $\alpha(x) = y$. It follows from $y <_B b$ and from Proposition 3.4 that

$$x = \alpha^{-1}(y) <_A \alpha^{-1}(b) = a.$$

Hence, the element x is contained in the set A_a implying that the element $y = \alpha(x)$ is contained in the set $\alpha(A_a)$. □

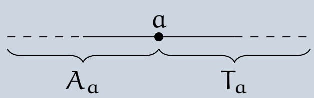
The following proposition describes the complement of an initial segment:

4.5 Proposition. *Let* A *be a totally ordered set, let* a *be an element of the set* A*, and let*

$$A_a := \{x \in A \mid x < a\} \text{ (initial segment) and } T_a := \{x \in A \mid x \geqslant a\}.$$

(a) We have $A = A_a \mathbin{\dot\cup} T_a$*.*

(b) We have $(A_a)^c = T_a$
(complement of the set A_a *in the set* A*).*



(c) We have $(T_a)^c = A_a$ *(complement of the set* T_a *in the set* A*).*

Proof. Obviously, we have $A_a \cap T_a = \emptyset$. Given an element x of the set A, it follows from the assumption that the set A is totally ordered that

$$\text{either } x < a \text{ or } x \geqslant a, \text{ that is, either } x \in A_a \text{ or } x \in T_a.$$

It follows that $A = A_a \mathbin{\dot\cup} T_a$.

(b) and (c) follow from (a). □

5 Chains of Ordered Sets

Definition of Chains of Ordered Sets:

5.1 Definition. Let I be a totally ordered index set. A family $(A_i, \leqslant_i)_{i \in I}$ is called a **chain of ordered sets** if the following conditions are fulfilled:

(i) The family $(A_i)_{i \in I}$ is a chain, that is, we have $A_i \subseteq A_j$ whenever $i \leqslant j$.

(ii) For each two elements i and j of the set I such that $i \leqslant j$, the order $\leqslant_i$ on the set A_i

is induced by the order $\leqslant_j$ on the set A_j, that is, we have

$$x \leqslant_i y \text{ if and only if } x \leqslant_j y \text{ for all } x, y \in A_i.$$

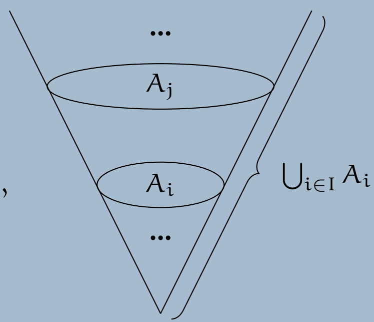
French / German. Chain of ordered sets = Chaîne d'ensembles ordonnés = Kette geordneter Mengen.

5.2 Theorem. *Let I be a totally ordered index set, let $(A_i, \leqslant_i)_{i \in I}$ be a chain of ordered sets, and let $A := \bigcup_{i \in I} A_i$.*

(a) There exists exactly one order $\leqslant_A$ on the set A such that for all elements i of the set I we have

$$x \leqslant_A y \text{ if and only if } x \leqslant_i y \text{ for all } x, y \in A_i,$$

that is, the orders $\leqslant_i$ are induced by the order $\leqslant$ for all elements i of the set I.



(b) If we have $A = A_i$ for an element i of the set I, then we have $(A, \leqslant_A) = (A, \leqslant_i)$.

Proof. (a) The existence of the set $A = \bigcup_{i \in I} A_i$ follows from the axiom of unions (for more details see Units *Unions and Intersections of Sets* [Garden 2020b] and *Families and the Axiom of Choice* [Garden 2020e]).

Let x and y be two elements of the set A. Since the family $(A_i)_{i \in I}$ is a chain of ordered sets, there exists an element $j = j_{x,y}$ of the set I such that the elements x and y are contained in the set A_j. Set $x \leqslant_A y$ if and only if $x \leqslant_j y$.

Let k be an element of the set I such that the elements x and y are contained in the set A_k. Since the family $(A_i, \leqslant_i)_{i \in I}$ is a chain of ordered sets, it follows that

$$x \leqslant_A y \text{ if and only if } x \leqslant_k y \text{ for all } x, y \in A_k.$$

In particular, it easily follows that the relation $\leqslant$ is an order.

For the uniqueness of the order $\leqslant_A$, let $\leqslant'$ be a second order on the set A fulfilling the above conditions. If x and y are two elements of the set A, there exists an element i of the set I such that the elements x and y are both contained in the set A_i. It follows that

$$x \leqslant_A y \text{ if and only if } x \leqslant_i y \text{ if and only if } x \leqslant' y, \text{ that is,}$$

$$(x, y) \in \leqslant_A \text{ if and only if } (x, y) \in \leqslant', \text{ that is, } \leqslant_A = \leqslant'.$$

(b) follows from (a). □

5.3 Remark. Theorem 5.2 allows to replace a chain of ordered sets by *one* ordered set.

5.4 Definition. Let I be a totally ordered index set, let $(A_i, \leqslant_i)_{i \in I}$ be a chain of ordered sets, and let $A := \bigcup_{i \in I} A_i$.

The order $\leqslant_A$ on the set A defined in Theorem 5.2 is called the **order of the set**

$A = \bigcup_{i \in I} A_i$ **induced by the chain of ordered sets** $(A_i, \leqslant_i)_{i \in I}$.

French / German. Order induced by a chain of ordered sets = Ordre induit par une chaîne d'ensembles ordonnés = Durch eine Kette geordneter Mengen induzierte Ordnung.

The following proposition prepares the proof of Theorem 5.6.

5.5 Proposition. *Let* $(A, \leqslant_A)$, $(B, \leqslant_B)$ *and* $(C, \leqslant_C)$ *be three ordered sets fulfilling the following conditions:*

(i) The set A *is a subset of the set* B, *and the set* B *is a subset of the set* C.

(ii) The order $\leqslant_A$ *is induced by the order* $\leqslant_B$, *that is, we have*

$$x \leqslant_A y \text{ if and only if } x \leqslant_B y \text{ for all } x, y \in A.$$

(iii) The order $\leqslant_B$ *is induced by the order* $\leqslant_C$, *that is, we have*

$$x \leqslant_B y \text{ if and only if } x \leqslant_C y \text{ for all } x, y \in B.$$

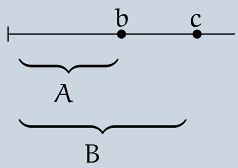
(a) The order $\leqslant_A$ *is induced by the order* $\leqslant_C$, *that is, we have*

$$x \leqslant_A y \text{ if and only if } x \leqslant_C y \text{ for all } x, y \in A.$$

(b) If there exist two elements b *and* c *of the sets* B *and* C, *respectively, such that*

$$\begin{aligned} A &= \{x \in B \mid x <_B b\} \text{ and} \\ B &= \{x \in C \mid x <_C c\}, \end{aligned}$$

then we have $A = \{x \in C \mid x <_C b\}$.



Proof. Set $I := \{A, B, C\}$ and define a total order $\leqslant$ on the set I by setting $A \leqslant B$, $B \leqslant C$ and $A \leqslant C$. Then the three ordered sets $(A, \leqslant_A)$, $(B, \leqslant_B)$ and $(C, \leqslant_C)$ form a chain of ordered sets. By Theorem 5.2, the order $\leqslant_C$ is the order on the set $C = A \cup B \cup C$ induced by this chain of ordered sets.

(a) In particular, we have

$$x \leqslant_A y \text{ if and only if } x \leqslant_C y \text{ for all } x, y \in A.$$

(b) It follows that $A = \{x \in B \mid x <_C b\}$. In order to show that $A = \{x \in C \mid x <_C b\}$, it remains to show that the set $\{x \in C \mid x <_C b\}$ is a subset of the set B:

Since $b <_C c$, we have

$$\{x \in C \mid x <_C b\} \subseteq \{x \in C \mid x <_C c\} = B.$$

□

5.6 Theorem. *Let* I *be a totally ordered index set, and let* $(A_i, \leqslant_i)_{i \in I}$ *be a chain of ordered sets fulfilling the following condition:*

If i *and* j *are two elements of the set* I *such that* $i < j$, *then we have* $(A_i, \leqslant_i) = (A_j, \leqslant_j)$, *or there exists an element* b_{ij} *of the set* A_j *such that* $A_i = \{x \in A_j \mid x <_j b_{ij}\}$, *that is, the set* A_i *is an initial segment of the set* A_j.

Let $A := \bigcup_{i\in I} A_i$, and let $\leqslant$ be the order on the set A induced by the chain $(A_i, \leqslant_i)_{i\in I}$ of ordered sets. Then for each element i of the set I, one of the following possibilities occurs:

(i) We have $(A_i, \leqslant_i) = (A, \leqslant)$.

(ii) There exists an element b_i of the set A such that $A_i = \{x \in A \mid x < b_i\}$.

In other words, for each element i of the set I, the set A_i is either an initial segment of the set A, or we have $A_i = A$.

Proof. Let i be an element of the set I. If $A_i = A$, it follows that $(A_i, \leqslant_i) = (A, \leqslant_A)$. This is Case (i).

So we may suppose that $A_i \neq A$. Then there exists an element y of the set $A \setminus A_i$. It follows that there exists an element j of the set I such that the set A_j contains the element y. Hence, there exists an element b_{ij} of the set A_j such that

$$A_i = \{x \in A_j \mid x <_j b_{ij}\} = \{x \in A_j \mid x < b_{ij}\}.$$

We claim that $A_i = \{x \in A \mid x < b_{ij}\}$: Since the set A_j is a subset of the set A, we have

$$A_i = \{x \in A_j \mid x < b_{ij}\} \subseteq \{x \in A \mid x < b_{ij}\}.$$

In order to show that the set $\{x \in A \mid x < b_{ij}\}$ is a subset of the set $A_i = \{x \in A_j \mid x < b_{ij}\}$, let z be an element of the set A such that $z < b_{ij}$. Assume that the element z is not contained in the set A_j. Then there exists an element k of the set I such that the set A_k contains the element z. Hence, there exists an element b_{jk} of the set A_k such that

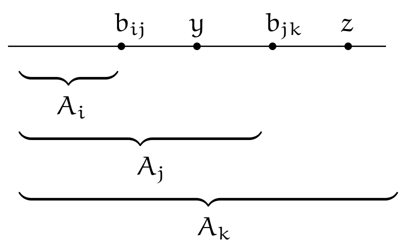
$$A_j = \{x \in A_k \mid x <_k b_{jk}\} = \{x \in A_k \mid x < b_{jk}\}.$$

Note that the element b_{ij} is contained in the set A_j implying the $b_{ij} < b_{jk}$.

Since the element z is contained in the set $A_k \setminus A_j$, it follows that

$$z \geqslant b_{jk} > b_{ij},$$

in contradiction to $z < b_{ij}$. $\square$



5.7 Theorem. *Let I be a totally ordered index set, and let $\big((A_i, \leqslant_{A_i})\big)_{i\in I}$ and $\big((B_i, \leqslant_{B_i})\big)_{i\in I}$ be two families of ordered sets with the following properties:*

(i) The families $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$ are chains of ordered sets (see Definition 5.1).

(ii) For each element i of the set I, there exists an isomorphism $\alpha_i : A_i \to B_i$.

(iii) For each two elements i and j of the set I such that $i \leqslant j$, we have $\alpha_j|_{A_i} = \alpha_i$, that is, we have

$$\alpha_j(x) = \alpha_i(x) \text{ for all } x \in A_i.$$

Let

$$A := \bigcup_{i\in I} A_i \text{ and } B := \bigcup_{i\in I} B_i,$$

and let $\leqslant_A$ and $\leqslant_B$ be the orders on the sets A and B induced by the chains of ordered sets $(A_i)_{i\in I}$ and $(B_i)_{i\in I}$, respectively.

Then there exists exactly one isomorphism

$$\alpha : (A, \leqslant_A) \to (B, \leqslant_B)$$

from the ordered set A *onto the ordered set* B *such that* $\alpha_i = \alpha|_{A_i}$*, that is,*

$$\alpha(x) = \alpha_i(x) \text{ for all } x \in A_i \text{ for all } i \in I.$$

Proof. By Theorem 2.18, there exists exactly one bijective function

$$\alpha : \bigcup_{i \in I} A_i \to \bigcup_{i \in I} B_i$$

with the property that $\alpha|A_i = \alpha_i$ for all elements i of the set I.

Let x and y be two elements of the set $\bigcup_{i \in I} A_i$. By Assumptions (i) and (ii), there exists an element $i = i_{x,y}$ of the set I such that the elements x and y are contained in the set A_i and an isomorphism $\alpha_i : A_i \to B_i$ from the set A_i onto the set B_i. It follows that

$$x \leqslant_A y \text{ if and only if } \alpha_i(x) \leqslant_{B_i} \alpha_i(y) \text{ if and only if } \alpha(x) \leqslant_B \alpha(y).$$

Hence, the function $\alpha : (A, \leqslant_A) \to (B, \leqslant_B)$ is an isomorphism. □

6 The Lemma of Zorn

We will prove the Lemma of Zorn as a consequence of the axiom of choice. The axiom of choice is explained in Unit *Families and the Axiom of Choice* [Garden 2020e]. We recall the following consequence of the axiom of choice which will be used in the proof of the Lemma of Zorn:

6.1 Theorem. *Let* S *be a non-empty set, and let* $\mathcal{P}(S)$ *be its power set, that is, the set of the subsets of the set* S*. Then there exists a function*

$$f : \mathcal{P}(S) \setminus \{\emptyset\} \to S$$

from the set of the non-empty subsets of the set S *into the set* S *such that the element* $f(X)$ *is contained in the set* X *for all non-empty subsets* X *of the set* S*.*

Proof. See Unit *Families and the Axiom of Choice* [Garden 2020e]. □

6.2 Theorem. (Lemma of Zorn - Special Case) *Let* S *be a non-empty set, and consider the ordered set* $\mathcal{P}(S) = \big(\mathcal{P}(S), \subseteq\big)$ *where* $\mathcal{P}(S)$ *denotes the power set of the set* S*. Let*

$$\emptyset \neq \mathcal{D} \subseteq \mathcal{P}(S)$$

be a non-empty set of subsets of the set S *such that for each chain* $\mathcal{C}$ *in the set* $\mathcal{D}$ *the set*

$$Z := \bigcup_{X \in \mathcal{C}} X$$

is also an element of the set $\mathcal{D}$*.*

Then the set $\mathcal{D} = (\mathcal{D}, \subseteq)$ contains a maximal element.

Proof. We will first define an auxiliary set $\mathcal{E}$ with the additional property that every subset B of a set A of the set $\mathcal{E}$ is contained in the set $\mathcal{E}$ (Step 1 and Step 2). Then we will show that the set $\mathcal{E}$ has a maximal element (Step 15). Finally, in Step 16 we will see that this maximal element is also a maximal element of the set $\mathcal{D}$.

Step 1. *Definition of the set $\mathcal{E}$:*

Let

$$\mathcal{E} := \mathcal{D} \cup \bigcup_{X \in \mathcal{D}} \mathcal{P}(X).$$

Step 2. *Let A be a set of the set $\mathcal{E}$, and let B be a subset of the set A. Then the set B is also contained in the set $\mathcal{E}$:*

If the set A is contained in the set $\mathcal{D}$, then the set B is contained in the set $\mathcal{P}(A)$ and therefore also contained in the set $\mathcal{E}$. If the set A is contained in the set $\mathcal{P}(X)$ for some set X of the set $\mathcal{D}$, then the set B is also contained in the set $\mathcal{P}(X)$ and therefore also contained in the set $\mathcal{E}$.

Step 3. *The empty set $\emptyset$ is an element of the set $\mathcal{E}$:*

Since the set $\mathcal{D}$ is non-empty, there exists an element A of the set $\mathcal{D}$. It follows from Step 2 that the empty set $\emptyset$ is an element of the set $\mathcal{E}$.

Step 4. *Let $\mathcal{C}$ be a chain in the set $\mathcal{E}$. Then we have*

$$\bigcup_{X \in \mathcal{C}} X \in \mathcal{E}:$$

By construction of the set $\mathcal{E}$, each set X of the chain $\mathcal{C}$ is a subset of a set Y_X of the set $\mathcal{D}$. It follows that

$$\bigcup_{X \in \mathcal{C}} X \subseteq \bigcup_{X \in \mathcal{C}} Y_X \in \mathcal{D} \subseteq \mathcal{E}.$$

It follows from Step 2 that the set $\bigcup_{X \in \mathcal{C}} X$ is contained in the set $\mathcal{E}$.

Step 5. *Definition of a function $g : \mathcal{E} \to \mathcal{E}$:*

By Theorem 6.1, there exists a function

$$f : \mathcal{P}(S) \setminus \{\emptyset\} \to S$$

from the set of the non-empty subsets of the set S into the set S such that

$$f(X) \in X \text{ for all } X \in \mathcal{P}(S) \setminus \{\emptyset\}.$$

For a set A of the set $\mathcal{E}$ let

$$\bar{A} := \{x \in S \mid A \cup \{x\} \in \mathcal{E}\}.$$

Obviously, the set $\bar{A}$ is an element of the power set $\mathcal{P}(S)$, and we have

$$A \subseteq \bar{A} \text{ for all } A \in \mathcal{E}.$$

In particular, we have $\bar{A} \neq \emptyset$ if $A \neq \emptyset$.

Let A be an element of the set $\mathcal{E}$ such that the set A is a proper subset of the set $\bar{A}$. Then we have $\bar{A} \setminus A \neq \emptyset$, and the element

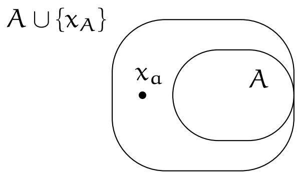
$$x_A := f(\bar{A} \setminus A) \in \bar{A} \setminus A$$

exists. Note that we have

$$A \subset A \cup \{x_A\} \text{ and } (A \cup \{x_A\}) \in \mathcal{E} \text{ for all } A \in \mathcal{E} \text{ with } A \subset \bar{A}.$$

We define the function $g : \mathcal{E} \to \mathcal{E}$ as follows:

$$g(A) := \begin{cases} A & \text{if} \quad A = \bar{A} \\ A \cup \{x_A\} & \text{if} \quad A \neq \bar{A}. \end{cases}$$



Step 6. *Let A be an element of the set $\mathcal{E}$. Then the set A is a maximal element of the set $\mathcal{E}$ if and only if $g(A) = A$:*

$\Rightarrow$: Suppose that the set A is a maximal element of the set $\mathcal{E}$. Assume that $g(A) \neq A$. Then the set $g(A) = A \cup \{x_A\}$ is an element of the set $\mathcal{E}$ such that $A \subset g(A)$, in contradiction to the maximality of the set A.

$\Leftarrow$: Suppose that the set A of the set $\mathcal{E}$ fulfills the condition $g(A) = A$. Assume that the set A is not maximal in the set $\mathcal{E}$. Then there exists an element B of the set $\mathcal{E}$ such that the set A is a proper subset of the set B. Let x be an element of the set $B \setminus A$. It follows from Step 2 that the set

$$A \cup \{x\} \subseteq B$$

is an element of the set $\mathcal{E}$. Hence, the element x is contained in the set $\bar{A}$ (Step 5). It follows that $A \neq \bar{A}$ and therefore $g(A) \neq A$, a contradiction.

Step 7. *Definition of a tower:*

A subset $\mathcal{T}$ of the set $\mathcal{E}$ is called a **tower** if it fulfills the following conditions:

(T1) We have $\emptyset \in \mathcal{T}$.

(T2) If A is an element of the set $\mathcal{T}$, then $g(A)$ is also an element of the set $\mathcal{T}$.

(T3) If $\mathcal{C}$ is a chain in the set $\mathcal{T}$, then the set $Z := \bigcup_{X \in \mathcal{C}} X$ is also an element of the set $\mathcal{T}$.

Step 7. *The set $\mathcal{E}$ is a tower:*

(T1): By Step 3, the empty set $\emptyset$ is an element of the set $\mathcal{E}$.

(T2): In Step 5 we have seen that the set $g(A)$ is contained in the set $\mathcal{E}$ for all elements A of the set $\mathcal{E}$.

(T3): Condition (T3) follows from Step 4.

Step 9. *Let $(\mathcal{T}_i)_{i \in I}$ be a family of towers for some index set I. Then the set*

$$\mathcal{T} := \bigcap_{i \in I} \mathcal{T}_i$$

is also a tower:

(T1): Since the empty set $\emptyset$ is an element of every tower $\mathcal{T}_i$, the empty set is also an element of the set $\mathcal{T}$.

(T2): Let A be an element of the set $\mathcal{T}$. Then the set A is an element of every tower $\mathcal{T}_i$. It follows from Condition (T2) that the set $g(A)$ is an element of every tower $\mathcal{T}_i$ implying that the set $g(A)$ is an element of the set $\mathcal{T}$.

(T3): Let $\mathcal{C}$ be a chain in the set $\mathcal{T}$. Then the chain $\mathcal{C}$ is also a chain in each tower $\mathcal{T}_i$. It follows from Condition (T3) that the set $Z := \bigcup_{X \in \mathcal{C}} X$ is an element of every tower $\mathcal{T}_i$ implying that the set Z is an element of the set $\mathcal{T}$.

Step 10. *Definition of the tower* $\mathcal{T}_0$*:*

Set

$$\mathcal{T}_0 := \bigcap \{\mathcal{T} \mid \mathcal{T} \text{ is a tower of } \mathcal{E}\}.$$

By Step 8, the set $\mathcal{E}$ is a tower. Hence, the above intersection is well-defined. By Step 9, the set $\mathcal{T}_0$ is a tower. Note that the set $\mathcal{T}_0$ is the smallest tower of the set $\mathcal{E}$.

The aim of Step 11 to Step 15 is to show that the tower $\mathcal{T}_0$ is a chain.

Step 11. *Definition of a comparable set:*

A set C of the tower $\mathcal{T}_0$ is called a **comparable set** if we have

$$X \subseteq C \text{ or } C \subseteq X \text{ for all } X \in \mathcal{T}_0.$$

Step 12. *Let* C *be a comparable set, and let* A *be a set of the tower* $\mathcal{T}_0$ *which is a proper subset of the set* C*. Then the set* $g(A)$ *is a subset of the set* C*:*

Since $\mathcal{T}_0$ is a tower, it follows from Condition (T2) of Step 7 that the set $g(A)$ is an element of the tower $\mathcal{T}_0$. Since the set C is comparable, we have

$$g(A) \subseteq C \text{ or } C \subseteq g(A).$$

Assume that the set C is a proper subset of the set $g(A)$. Then we have

$$A \subset C \subset g(A).$$

Hence, there exists an element x of the set $C \setminus A$ and an element y of the set $g(A) \setminus C$. It follows that $x \neq y$ and that the set $\{x, y\}$ is a subset of the set $g(A) \setminus A$. On the other hand, by Step 5, we have $g(A) = A$ or $g(A) = A \cup \{x_A\}$ implying that

$$\{x, y\} \subseteq g(A) \setminus A = \emptyset \text{ or } \{x, y\} \subseteq g(A) \setminus A = \{x_A\},$$

a contradiction.

Step 13. *Let* C *be a comparable set, and let* A *be a set of the tower* $\mathcal{T}_0$*. Then we have*

$$A \subseteq C \text{ or } g(C) \subseteq A :$$

Let C be a comparable set, and set

$$\mathcal{A}_C := \{X \in \mathcal{T}_0 \mid X \subseteq C \text{ or } g(C) \subseteq X\}.$$

Step 13.1 *The set* $\mathcal{A}_C$ *is a tower:*

(T1): Since the empty set $\emptyset$ is an element of the tower $\mathcal{T}_0$ and since the empty set is a subset of the set C, the empty set is an element of the set $\mathcal{A}_C$.

(T2): Let Y be an element of the set $\mathcal{A}_C = \{X \in \mathcal{T}_0 \mid X \subseteq C \text{ or } g(C) \subseteq X\}$. Then one of the following three cases may occur:

$$Y \subset C, Y = C \text{ or } g(C) \subseteq Y.$$

Case 1. *Suppose that* $Y \subset C$.

It follows from Condition (T2) of Step 5 that the set $g(Y)$ is an element of the tower $\mathcal{T}_0$ and from Step 12 that the set $g(Y)$ is a subset of the set C implying that the set $g(Y)$ is an element of the set $\mathcal{A}_C$.

Case 2. *Suppose that* $Y = C$.

Then we have $g(Y) = g(C)$. It follows that the set $g(C)$ is a subset of the set $g(Y)$ implying that the set $g(Y)$ is an element of the set $\mathcal{A}_C$.

Case 3. *Suppose that* $g(C) \subseteq Y$.

Since the set Y is a subset of the set $g(Y)$, it follows that the set $g(C)$ is a subset of the set $g(Y)$ implying that the set $g(Y)$ is an element of the set $\mathcal{A}_C$.

(T3): Let $\mathcal{C}$ be a chain in the set $\mathcal{A}_C$, and let $Z := \bigcup_{X \in \mathcal{C}} X$. By Condition (T3) of Step 5, the set Z is an element of the tower $\mathcal{T}_0$. Since

$$\mathcal{C} \subseteq \mathcal{A}_C = \{X \in \mathcal{T}_0 \mid X \subseteq C \text{ or } g(C) \subseteq X\},$$

we may distinguish the following two cases:

Case 1. *There exists an element* A_0 *in the chain* $\mathcal{C}$ *such that the set* $g(C)$ *is a subset of the set* A_0.

Then we have

$$g(C) \subseteq A_0 \subseteq \bigcup_{X \in \mathcal{C}} X = Z$$

implying that the set Z is an element of the set $\mathcal{A}_C$.

Case 2. *Each set* X *of the chain* $\mathcal{C}$ *is a subset of the set* C.

It follows that

$$Z = \bigcup_{X \in \mathcal{C}} X \subseteq C$$

implying that the set Z is an element of the set $\mathcal{A}_C$.

Step 13.2 *We have* $\mathcal{A}_C = \mathcal{T}_0$:

Since $\mathcal{T}_0$ is the smallest tower in the set $\mathcal{D}$ (Step 8) and since, by Step 11.1, the tower $\mathcal{A}_C$ is a subset of the tower $\mathcal{T}_0$, it follows that $\mathcal{A}_C = \mathcal{T}_0$.

Step 13.3. *Let* C *be a comparable set, and let* A *be a set of the tower* $\mathcal{T}_0$. *Then we have*

$$A \subseteq C \textit{ or } g(C) \subseteq A:$$

Let C be a comparable set. The assertion follows from the fact that

$$\mathcal{T}_0 = \mathcal{A}_C = \{X \in \mathcal{T}_0 \mid X \subseteq C \text{ or } g(C) \subseteq X\}.$$

Step 14. *Let* C *be a comparable set of the tower* $\mathcal{T}_0$. *Then the set* $g(C)$ *is also comparable:*

Note that, by Condition (T2) of Step 5, the set $g(C)$ is an element of the tower $\mathcal{T}_0$. Let A be a set of the tower $\mathcal{T}_0$. We have to show that

$$A \subseteq g(C) \text{ or } g(C) \subseteq A:$$

By Step 13, the set A is either a subset of the set C or the set $g(C)$ is a subset of the set A. If the set $g(C)$ is a subset of the set A, there is nothing to show.

So suppose that the set A is a subset of the set C. Since the set C is a subset of the set $g(C)$ (Step 5), it follows that the set A is a subset of the set $g(C)$.

Step 15. *The tower* $\mathcal{T}_0$ *is a chain:*

Step 15.1 *Let* $\mathcal{C}_0 := \{C \in \mathcal{T}_0 \mid C$ *is comparable*$\}$. *Then the set* $\mathcal{C}_0$ *is a tower:*

(T1): Since the empty set $\emptyset$ is comparable (we have $\emptyset \subseteq A$ for all elements A of the tower $\mathcal{T}_0$), the empty set is an element of the set $\mathcal{C}_0$.

(T2): Let C be a comparable set. By Step 14, the set $g(C)$ is also comparable.

(T3): Let $\mathcal{C}$ be a chain in the set $\mathcal{C}_0$, and let $Z := \bigcup_{A \in \mathcal{C}} A$. We have to show that the set Z is comparable. For, let A be a set of the tower $\mathcal{T}_0$. We have to show that $A \subseteq Z$ or $Z \subseteq A$:

Since the chain $\mathcal{C}$ consists of comparable sets, we have $A \subseteq C$ or $C \subseteq A$ for all elements C of the chain $\mathcal{C}$. Therefore we may distinguish the following two cases:

Case 1. *There exists an element C_0 in the chain $\mathcal{C}$ such that $A \subseteq C_0$.*

Then we have

$$A \subseteq C_0 \subseteq \bigcup_{X \in \mathcal{C}} X = Z.$$

Case 2. *The set C is a subset of the set A for each set A of the chain $\mathcal{C}$.*

It follows that

$$Z = \bigcup_{C \in \mathcal{C}} C \subseteq A.$$

It follows from Case 1 and Case 2 that we have

$$A \subseteq Z \text{ or } Z \subseteq A \text{ for all } A \in \mathcal{T}_0.$$

Hence, the set Z is an element of the set $\mathcal{C}_0$.

Step 15.2 *We have $\mathcal{C}_0 = \mathcal{T}_0$:*

Since $\mathcal{T}_0$ is the smallest tower in the set $\mathcal{D}$ (Step 8) and since, by Step 15.1, the tower $\mathcal{C}_0$ is a subset of the tower $\mathcal{T}_0$, it follows that $\mathcal{C}_0 = \mathcal{T}_0$.

Step 15.3 *The tower $\mathcal{T}_0$ is a chain:*

Let A and B be two subsets of the tower $\mathcal{T}_0$. It follows from Step 15.2 that the sets A and B are comparable. In particular, we have $A \subseteq B$ or $B \subseteq A$, that is, the tower $\mathcal{T}_0$ is a chain.

Step 16. *Let $Z := \bigcup_{A \in \mathcal{T}_0} A$. Then the set Z is a maximal element of the set $\mathcal{E}$:*

Step 16.1 *We have $g(Z) = Z$:*

Since the set $\mathcal{T}_0$ is a chain (Step 15) and since the set $\mathcal{T}_0$ is a tower (Step 10), it follows from Condition (T3) of Step 7 that the set Z is an element of the tower $\mathcal{T}_0$. By Condition (T2) of Step 7, the set $g(Z)$ is also an element of the tower $\mathcal{T}_0$. It follows that

$$g(Z) \subseteq \bigcup_{A \in \mathcal{T}_0} A = Z.$$

On the other hand, the set Z is a subset of the set $g(Z)$ (Step 5) implying that $g(Z) = Z$.

Step 16.2 *The set Z is a maximal element of the set $\mathcal{E}$:*

The assertion follows from Step 16.1 and Step 6.

Step 17. *The set Z is a maximal element of the set $\mathcal{D}$:*

Step 17.1 *The set Z is contained in the set $\mathcal{D}$:*

Since $\mathcal{E} = \mathcal{D} \cup \bigcup_{X \in \mathcal{D}} \mathcal{P}(X)$, we may assume that the set Z is contained in the set $\bigcup_{X \in \mathcal{D}} \mathcal{P}(X)$. Assume that the set Z is not contained in the set $\mathcal{D}$. Then there exists a set X of the set $\mathcal{D}$ such that the set Z is strictly contained in the set X, in contradiction to the fact that the set Z is maximal in $\mathcal{E}$.

Step 17.2 *The set Z is a maximal element of the set $\mathcal{D}$:*

Since the set $\mathcal{D}$ is a subset of the set $\mathcal{E}$, since the set Z is maximal in $\mathcal{E}$ and since the set Z is contained in the set $\mathcal{D}$ (Step 17.1), the set Z is maximal in $\mathcal{D}$. □

6.3 Theorem. (Lemma of Zorn - General Case) *Let $\mathcal{A} = (A, \leqslant)$ be an ordered set such that every chain C of the set $\mathcal{A}$ has an upper bound in the set $\mathcal{A}$.*

Then the set $\mathcal{A}$ contains a maximal element.

Proof. We shall apply Theorem 6.2. For, we will define a set $\mathcal{D}$, and we will show that the set $\mathcal{D}$ fulfills the assumptions of Theorem 6.2.

Step 1. *Definition of the set $\mathcal{D}$*:

Let

$$\mathcal{D} := \{X \subseteq A \mid X \text{ is a chain in } \mathcal{A}\}.$$

(More formally, the pair $(X, \leqslant)$ with the induced order is a chain in the set $\mathcal{A}$.)

Step 2. *Let D be an element of the set $\mathcal{D}$.*

Then there exists an element z_D of the set A such that

$$x \leqslant z_D \text{ for all } x \in D:$$

Since the set D is a chain in the set $\mathcal{A}$, it follows that the set D has an upper bound z_D in the set $\mathcal{A}$. The assertion follows.

Step 3. *Let $\mathcal{C}$ be a chain in the set $\mathcal{D} = (\mathcal{D}, \subseteq)$.*

Then the set

$$Z := \bigcup_{C \in \mathcal{C}} C$$

is an element of the set $\mathcal{D}$:

We have to show that the set Z is a chain in the set $\mathcal{A}$: Since every set C of the chain $\mathcal{C}$ is a subset of the set A, the set $Z := \bigcup_{C \in \mathcal{C}} C$ is also a subset of the set A.

Let x and y be two elements of the set Z. Then there exist two sets C_x and C_y of the chain $\mathcal{C}$ containing the elements x and y, respectively. Since the set $\mathcal{C}$ is a chain in the set $\mathcal{D}$, we have $C_x \subseteq C_y$ or $C_y \subseteq C_x$. W.l.o.g. suppose that $C_x \subseteq C_y$. Since the set C_y is a chain in the set $\mathcal{A}$ containing the elements x and y, it follows that $x \leqslant y$ or $y \leqslant x$ implying that the set Z is a chain in the set $\mathcal{A}$.

Step 5. *There exists a maximal element D in the set $\mathcal{D} = (\mathcal{D}, \subseteq)$:*

The assertion follows from Step 3, Step 4 and Theorem 6.2.

Step 6. *There exists a maximal element in the set $\mathcal{A}$:*

By Step 5, there exists a maximal element D in the set $\mathcal{D}$. By Step 2, there exists an element z_D of the set D such that

$$x \leqslant z_D \text{ for all } x \in D.$$

We claim that the element z_D is a maximal element of the set $\mathcal{A}$: For, assume that there exists an element y of the set A such that $z_D < y$. Since the element z_D is a maximal element of the set D, it follows from that $z_D < y$ that the element y is not contained in the set D.

Note that it follows from $x \leqslant z_D$ for all elements x of the set D and from $z_D < y$ that $x < y$ for all elements x of the set D.

Let $E := D \cup \{y\}$. Obviously, the set E is a subset of the set A. Since the set D is a chain in the set A and since $x \leqslant y$ for all elements x of the set D, it follows that the set $E = D \cup \{y\}$ is also a chain. In particular, the set E is an element of the set $\mathcal{D}$, in contradiction to the fact that the set D is a maximal element of the set $\mathcal{D}$ and that the set D is a proper subset of the set E. □

6.4 Remark. The axiom of choice and the Lemma of Zorn are equivalent in the following sense: The set theoretical axiomatic of Zermelo and Fraenkel consists of the following axioms:

ZFC-0:	Basic Axiom
ZFC-1:	Axiom of Extension
ZFC-2:	Axiom of Existence
ZFC-3:	Axiom of Specification
ZFC-4:	Axiom of Foundation
ZFC-5:	Axiom of Pairing
ZFC-6:	Axiom of Unions
ZFC-7:	Axiom of Powers
ZFC-8:	Axiom of Substitution
ZFC-9:	Axiom of Choice
ZFC-10:	Axiom of Infinity

These axioms are explained in the units *The Mathematical Universe* [Garden 2020a] (ZFC-0 to ZFC-4), *Unions and Intersections of Sets* [Garden 2020b] (ZFC-5 to ZFC-7), *Families and the Axiom of Choice* [Garden 2020e] (ZFC-8 and ZFC-9) and *Successor Sets and the Axioms of Peano* [Garden 2020f] (ZFC-10).

In Theorem 6.3 we have deduced the Lemma of Zorn from Axioms ZFC-0 to ZFC-9 (in fact, we did not make use of the axiom of substitution (Axiom ZFC-8)). Alternatively, we could have chosen the Lemma of Zorn as Axiom ZFC-9, and we could have deduced the axiom of choice from the Lemma of Zorn and the preceding axioms.

In Unit *Well Ordered Sets* [**nst-well-ordered-sets**] we will show that every set can be endowed with a well ordering. This theorem is again equivalent to the axiom of choice. In Unit *Well Ordered Sets* we will prove the theorem about well orderings using the Lemma of Zorn, and we will show that the axiom of choice can be deduced from the theorem about well orderings. So, finally, we will obtain the following chain of conclusions:

Axiom of Choice $\Rightarrow$ Lemma of Zorn $\Rightarrow$ Theorem about Well Orderings $\Rightarrow$ Axiom of Choice.

Historical Notes:

The Lemma of Zorn has been formulated by Max Zorn in 1935 as follows:

Definition 1. *A set $\mathfrak{B} = \{B\}$ of sets B is called a* **chain** *if for every two sets B_1, B_2 either $B_1 \supset B_2$ or $B_2 \supset B_1$.*

Definition 2. *A set $\mathfrak{A}$ of sets A is said to be* **closed (right-closed)** *if it contains the union $\bigcup_{B \in \mathfrak{B}} B$ of every chain $\mathfrak{B}$ contained in $\mathfrak{A}$.*

Then our maximum principle is expressible in the following form.

(MP) *In a closed set $\mathfrak{A}$ of sets A there exists at least one, A^*, not contained as a proper subset in any other $A \in \mathfrak{A}$.*

See [Zorn 1935, p. 667].

Curiously, the article [Zorn 1935] does not contain a proof of Zorn's lemma, but Zorn announced its proof in a later publication:

In another paper I shall discuss the relations between MP, the axiom of choice, and the well-ordering theorem. I shall show that they are equivalent if the axiom yielding the set of all subsets of a set is available.

See [Zorn 1935, p. 669].

However, this paper has never been written. Instead, the article [Zorn 1935] contains some applications of the Lemma of Zorn such as the existence of maximal ideals in a unitary ring.

Since the ideas for a proof of the Lemma of Zorn are already contained in the two papers of Ernst Zermelo about the well-ordering theorem ([Zermelo 1904] and [Zermelo 1908]), Oliver Deiser [Deiser 2020, p. 267] suggests to call the Lemma of Zorn the Lemma of Zermelo-Zorn.

Sometimes the Lemma of Zorn is also called the Lemma of Kuratowski-Zorn (see for example Wikipedia (Lemma of Zorn) (Version from 28. April 2020) due to the article [Kuratowski 1922] of Casimir Kuratowski. Kuratowski's result is related to the Lemma of Zorn, but, in fact, it is not exactly the maximum principle of Zorn explained above. Nevertheless, it is a general method to avoid transfinite numbers in the proof of theorems like the well-ordering theorem.

The result close to the Lemma of Zorn is Theorem II' of [Kuratowski 1922]. Theorem II' reads as follows:

Théorème II'. *Si la fonction* $G(X)$ *satisfait aux conditions (5) et (21), l'ensemble* $S(A)$ *(c'est à dire, le plus grand ensemble de la classe* $N(A)$*) est le plus petit sous-ensemble* Z *de* E *qui, tout en contenant* A*, satisfait à l'égalité (15).*

See [Kuratowski 1922, p. 83].

Theorem II'. *If the function* $G(X)$ *satisfies conditions (5) et (21), the set* $S(A)$ *(that is, the biggest set of class* $N(A)$*) is the smallest subset* Z *of* E *which, containing* A*, satisfies inequality (15).*

(Translation by the author.)

The details of Theorem II' are spread throughout the article. For the reader's convenience we reformulate Theorem II' as follows:

Theorem. (Theorem of Kuratowski) *Let* E *be a non-empty set, and let*

$$G : \mathcal{P}(E) \to \mathcal{P}(E)$$

be a function from the power set of the set E *into itself with the following additional properties:*

$$X \subseteq G(X) \text{ for all } X \in \mathcal{P}(E) \text{ and}$$

$$X \subseteq Y \text{ implies } G(X) \subseteq G(Y) \text{ for all } X, Y \in \mathcal{P}(E).$$

If A *is a subset of the set* E*, then there exists a smallest set* $S(A)$ *of the power set* $\mathcal{P}(E)$ *containing the set* A *as a subset such that*

$$G\big(S(A)\big) = S(A).$$

Smallest means that if Z *is a further element of the power set* $\mathcal{P}(E)$ *such that*

$$A \subseteq Z \text{ and } G(Z) = Z,$$

then the set $S(A)$ *is a subset of the set* Z.

Conditions (5), (21) and (15) are the conditions $X \subseteq G(X)$, $X \subseteq Y \Rightarrow G(X) \subseteq G(Y)$ and $G\big(S(A)\big) = S(A)$, respectively. The class $N(A)$ is an auxiliary set in the proof of Kuratowski. In Theorem III Kuratowski shows that this set $N(A)$ is a chain. Kuratowski applies his method for an alternative proof of the well-ordering theorem and some other results.

7 Notes

We want to mention that there exists a film with the title Zorn's lemma from 1970 directed by Hollis Frampton. Fore more details see Wikipedia.

8 Literature

A list of text books about set theory can be found at Literature about Set Theory.

Cantor, Georg (1895). "Beiträge zur Begründung der transfiniten Mengenlehre". In: *Mathematische Annalen* 46, pp. 481–512 (cit. on p. 18).

Deiser, Oliver (2020). *Einführung in die Mengenlehre. Die Mengenlehre Cantors und ihre Axiomatisierung durch Ernst Zermelo*. URL: www.aleph1.info (visited on 03/14/2020). Earlier versions of this book have been published at Springer Verlag, Berlin, Heidelberg, New York. (Cit. on p. 32).

Hausdorff, Felix (1914). *Grundzüge der Mengenlehre*. Leipzig: Veit and Comp. For a new edition see [Hausdorff 2002]. (Cit. on pp. 14, 33).

— (2002). *Gesammelte Werke*. Vol. 2: *Grundzüge der Mengenlehre*. Ed. by Brieskorn E. et al. Berlin, Heidelberg, and New York: Springer Verlag. For the first edition see [Hausdorff 1914]. (Cit. on p. 33).

Kuratowski, Casimir (1922). "Une Méthode d'Elimination des Nombres Transfinis des Raisonnements Mathématiques". In: *Fundamenta Mathematicae* 3.1, pp. 76–108 (cit. on p. 32).

Zermelo, Ernst (1904). "Beweis, dass jede Menge wohlgeordnet werden kann". In: *Mathematische Annalen* 59, pp. 514–516 (cit. on p. 32).

— (1908). "Neuer Beweis für die Möglichkeit einer Wohlordnung". In: *Mathematische Annalen* 65, pp. 107–128 (cit. on p. 32).

Zorn, Max (1935). "A remark on Method in Transfinite Algebra". In: *Bulletin of the American Mathematical Society* 41.10, pp. 667–670 (cit. on p. 32).

9 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *The Mathematical Universe*. Version 1.0.2. URL: https://www.math-garden.com/unit/nst-universe#nst1-sec-download (cit. on p. 31).

— (2020b). *Unions and Intersections of Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-unions#nst-unions-download (cit. on pp. 21, 31).

Garden, M. (2020c). *Direct Products and Relations*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-direct-products#nst-product-download (cit. on pp. 3, 6–8).

— (2020d). *Functions and Equivalent Sets*. Version 1.0.1. URL: https://www.math-garden.com/unit/nst-functions#nst-functions-download (cit. on pp. 4, 11, 12, 16, 17).

— (2020e). *Families and the Axiom of Choice*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-families#nst-families-download (cit. on pp. 6, 12, 21, 24, 31).

— (2020f). *Successor Sets and the Axioms of Peano*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-successor-sets#nst-successor-sets-download (cit. on p. 31).

— (2021a). *Vector Spaces*. Version 1.0.0. In preparation (cit. on p. 6).

— (2020g). *Groups and Subgroups*. Version 1.0.0. In preparation (cit. on p. 17).

— (2021b). *Ideals in Rings*. Version 1.0.0. In preparation (cit. on p. 6).

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If you are willing to share comments and ideas to improve the present unit or hints about further references, we kindly ask you to send a mail to info@math-garden.com or to use the contact form on www.math-garden.com. Contributions are highly appreciated.

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