M. Garden

The Natural Numbers and The Principle of Induction

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1 Introduction

The present unit is part of the walk *The Axioms of Zermelo and Fraenkel*. It introduces the natural numbers in the context of the axiomatics of Zermelo and Fraenkel.

The Definition of the Natural Numbers (see Section 2):

What are the characteristic properties of the natural numbers which may serve as a basis for their formal definition? Peano [Peano 1889] gave in 1889 the following answer:

A set A has the characteristic properties of the set $\mathbb{N}_0$ of the natural numbers if it fulfills the following conditions:

(P1) The set A contains the number 0. The number 0 can be understood as the starting point for the definition of the natural numbers.

(P2) Starting with the number 0 Peano defines a successor $1 = 0^+$, then a successor $2 = 1^+$ and so on. More formally, he requires the existence of a function $+ : A \to A;\ x \mapsto x^+$ from the set A into itself. For a natural number n the successor n^+ has the meaning $n^+ = n + 1$.

(P3) We want that $n+1 \neq 0$ for all natural numbers n. More formally, we require that $x^+ \neq 0$ for all elements x of the set A.

(P4) We want that the mapping

$$+ : A \to A,\ + : x \mapsto x^+ (= x + 1)$$

is injective. In other words: If x and y are two elements of the set A such that $x^+ = y^+$, then we have $x = y$.

Note that Axioms (P1) to (P4) are not sufficient to guarantee that the mapping $+ : A \to A \setminus \{0\}$ is surjective. As an example we consider the set

$$A := \mathbb{N}_0 \cup \{(n, 1) \mid n \in \mathbb{N}_0\}$$

with

$$n^+ := n + 1 \text{ and } (n, 1)^+ := (n + 1, 1) \text{ for all } n \in \mathbb{N}_0.$$

It fulfills the axioms (P1) to (P4), but the element $(0, 1)$ does not have a predecessor, that is, there is no element $x \in A$ such that $x^+ = (0, 1)$. Peano solves this problem with an additional axiom (P5) as follows:

(P5) If B is a subset of the set A such that

$$0 \in B \text{ and } x^+ \in B \text{ for all } x \in B,$$

then we have $B = A$.

This axiom has the additional advantage that it provides the basis for the principle of induction.

The axioms (P1) to (P5) are called the **axioms of Peano** (see Definition 2.1), and we have shown in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c] that there exists exactly one set ω fulfilling the axioms of Peano and additionally the conditions

$$0 := \emptyset \text{ and } A^+ := A \cup \{A\} \text{ for all } A \in \omega.$$

See Theorem 2.2. The set ω is called the **minimal successor set**.

The formal definition of the set $\mathbb{N}_0$ of the natural numbers is now very easy: We just set

$$\mathbb{N}_0 := \omega;\ 0 := \emptyset \text{ and } n+1 := n^+ := n \cup \{n\} \text{ for all } n \in \mathbb{N}_0 = \omega.$$

See Definitions 2.3 and 2.4.

In Theorem 2.5 we shall see that the mapping $\mathbb{N}_0 \to \mathbb{N}_0 \setminus \{0\} = \mathbb{N}$, $n \mapsto n+1$ is indeed bijective.

The Principle of Induction and the Recursive Definition of Functions (see Section 3):

The principle of induction states that if an assertion A_0 is true and if we may deduce the assertion A_{n+1} from the assertion A_n for all natural numbers n, then the assertion A_n is true for all natural numbers n. Originally, this principle has been considered as a general proof technique. In the context of the axiomatics of Zermelo and Fraenkel it is just a theorem which is an immediate consequence of Axiom (P5) of Peano (see Theorem 3.1).

Closely related to the principle of induction is the possibility to define a function $\alpha : \mathbb{N}_0 \to X$ from the set of the natural numbers into a set X recursively. For example, we will define the exponentiation a^n recursively by

$$a^0 := 1 \text{ and } a^{n+1} := a^n \cdot a.$$

In other words, we have to construct a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ with the properties that

$$\alpha(0) = 1 \text{ and } \alpha(n+1) = \alpha(n) \cdot a.$$

Then we may define $a^n := \alpha(n)$. The formal procedure is as follows: We first define an auxiliary function $f : \mathbb{N}_0 \to \mathbb{N}_0$ by $f : n \mapsto n \cdot a$. Then Theorem 3.4 guarantees the existence and the uniqueness of a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0$ such that

$$\alpha(0) = 1 \text{ and } \alpha(n+1) = f\big(\alpha(n)\big) = \alpha(n) \cdot a \text{ for all } n \in \mathbb{N}_0.$$

In some cases this procedure has to be extended a little bit. Examples are discussed in Section 7.

The Addition of Natural Numbers (see Section 4):

A crucial property of the natural numbers is the possibility to add natural numbers. The sum $m + n$ is defined recursively (see Definition 4.2). The basic properties of addition are that addition is associative (Theorem 4.5), that addition is commutative (Theorem 4.7) and that the so-called cancellation rules hold (Theorem 4.8).

It is remarkable that the axiomatics of Zermelo and Fraenkel provides a possibility to *prove* these properties. Before that, one just had to accept these properties.

One way to define the natural order $0 < 1 < 2 < \ldots < n < n+1 < \ldots$ on the set of the natural numbers is based on the following property of the natural numbers: We will see in Theorem 4.9 that given two natural numbers m and n exactly one of the following cases occurs:

(i) We have $n = m$.

(ii) There exists a natural number $k \neq 0$ such that $n = m + k$.

(iii) There exists a natural number $k \neq 0$ such that $m = n + k$.

Case (ii) means $m < n$. Case (iii) means $n < m$. We will explain this in detail in Section 8.

The Multiplication of Natural Numbers (see Section 5):

A further crucial property of the natural numbers is the possibility to multiply natural numbers. The product $m \cdot n$ is defined recursively (see Definition 5.2). The basic properties of multiplication are that multiplication is associative (Theorem 5.6), that multiplication is commutative (Theorem 5.8) and that the so-called cancellation rules hold (Theorem 5.10).

The combination of addition and multiplication is expressed by the distributive laws (see Theorem 5.9).

The Power of Natural Numbers (see Section 6):

Finally, we consider the power of natural numbers. The power m^n is defined recursively (see Definition 6.2). The basic properties of exponentiation are

$$(km)^n = k^n \cdot m^n,\ k^{m+n} = k^m \cdot k^n \text{ and } (k^m)^n = k^{mn} \text{ for all } k, m, n \in \mathbb{N}_0.$$

See Theorem 6.5.

Factorial of n and the Fibonacci Numbers (see Section 7):

The factorial $n!$ of a natural number n and the Fibonacci numbers F_n are two further examples of recursively defined functions. They are defined as follows:

$$n! := 1 \cdot 2 \cdot \ldots \cdot n, \text{ or, equivalently, } 0! := 1 \text{ and } (n+1)! := n! \cdot (n+1) \text{ for all } n \in \mathbb{N}_0.$$

$$F_0 := 0,\ F_1 := 1 \text{ and } F_{n+2} := F_{n+1} + F_n \text{ for all } n \in \mathbb{N}_0.$$

Even though their definition also relies on Theorem 3.4, the process is a little bit more complicated. We will explain this in Examples 7.1 and 7.2.

The Standard Order on the Natural Numbers (see Section 8):

Given two natural numbers m and n we need a method to decide whether $m = n$, $m < n$ or $m > n$ with respect to the natural order $0 < 1 < 2 < \ldots n < n+1 < \ldots$. More precisely, we need a formal definition for this order. One approach is the observation that if $m > n$, then there exists a natural number $k \neq 0$ such that $m = n + k$, for example $5 = 3 + 2$. Given two natural numbers m and n according to Theorem 4.9 exactly one of the following possibilities occurs:

	Equation	Meaning
(i)	$m = n$	$m = n$
(ii)	$n = m + k,\ k \neq 0$	$m < n$
(iii)	$m = n + k,\ k \neq 0$	$m > n$

Hence, one may define $m \leqslant n$ if there exists a natural number k such that $n = m + k$ (see Definition 8.1). A first important conclusion is the fact that the pair $(\mathbb{N}_0, \leqslant)$ is a totally ordered set (see Theorem 8.2).

The definition of the natural order on the set of the natural numbers allows us to define the set

$$\{0, 1, \ldots, n\} := \{x \in \mathbb{N}_0 \mid 0 \leqslant x \leqslant n\} \text{ for all } n \in \mathbb{N}_0.$$

Each natural number is a the the same time a set. We recall that

$$0 = \emptyset,\ 1 = \{0\},\ 2 = \{0, 1\}, \ldots$$

The above notation allows us to give the following concise description of the set n: We have

$$0 = \emptyset \text{ and } n + 1 = \{0, 1, \ldots, n\} \text{ for all } n \in \mathbb{N}_0.$$

See Theorem 8.9. The order relation on the set $\mathbb{N}_0$ can also be expressed as follows:

$$m \leqslant n \text{ if and only if } m \subseteq n \text{ for all } m, n \in \mathbb{N}_0.$$

See Theorem 8.5 and Remark 8.6.

The next task is to explore the relation between the (natural) order on the set $\mathbb{N}_0$ and the algebraic operations addition, multiplication and exponentiation. The most important results are as follows: Let a, b, c, d, m, n and x be natural numbers. Then we have

$$\begin{aligned}
a \leqslant c \text{ and } b \leqslant d &\Rightarrow a + b \leqslant c + d \\
x + m \leqslant x + n &\Rightarrow m \leqslant n \\
a \leqslant c \text{ and } b \leqslant d &\Rightarrow ab \leqslant cd \\
x \neq 0 \text{ and } xm \leqslant xn &\Rightarrow m \leqslant n \\
x \leqslant y &\Rightarrow x^n \leqslant y^n \\
x \neq 0 \text{ and } m \leqslant n &\Rightarrow x^m \leqslant x^n \\
n \neq 0 \text{ and } x^n \leqslant y^n &\Rightarrow x \leqslant y \\
x \neq 0,\ x \neq 1 \text{ and } x^m \leqslant x^n &\Rightarrow m \leqslant n
\end{aligned}$$

Generalized Arithmetical Laws (see Section 9):

The additive associative law

$$(a + b) + c = a + (b + c)$$

allows us to write $a+b+c$ instead of $(a+b)+c$ or $a+(b+c)$. The sum $\sum_{j=1}^{n} x_j$ $(x_1, \ldots, x_n \in \mathbb{N}_0)$ is defined recursively by

$$\sum_{j=1}^{1} x_j := x_1 \text{ and } \sum_{j=1}^{n+1} x_j := \Big(\sum_{j=1}^{n} x_j\Big) + x_{n+1}.$$

This means that

$$\sum_{j=1}^{n} x_j = \big(\big((x_1 + x_2) + x_3\big) + \ldots + x_n\big).$$

The generalized associative law (see Proposition 9.4) allows us to omit all these brackets and to write

$$\sum_{j=1}^{n} x_j = x_1 + \ldots + x_n.$$

The same is true for the multiplication (see Proposition 9.4). In a similar way the generalized commutative law (see Proposition 9.6) and the generalized distributive laws (see Proposition 9.8) generalize the commutative law $x + y = y + x$ and the distributive laws $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$.

Dedekind's Construction of the Natural Numbers (see Section 10):

Richard Dedekind was the first to give an axiomatic definition of the natural numbers (see [Dedekind 1888]). Peano's work and the axioms of Peano are based on this work of Dedekind. In the present unit we did not follow the approach of Dedekind, but introduced the set of the natural numbers as the minimal successor set defined in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c].

In Section 10 we will explain the brilliant approach of Dedekind.

2 The Definition of the Natural Numbers

The definition of the natural numbers will be based on the so-called Peano sets explained in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c].

Peano Sets:

2.1 Definition. Let A be a set.

(a) The set A **fulfills the axioms of Peano** if it fulfills the following conditions:

(P1) The set A contains a distinguished element 0. In particular, the set A is not empty.

(P2) There exists a function $^+ : A \to A$, $x \mapsto x^+$ from the set A into itself.

(P3) We have $x^+ \neq 0$ for all elements x of the set A.

(P4) If x and y are two elements of the set A such that $x^+ = y^+$, then we have $x = y$, that is, the function $^+ : A \to A$ is injective.

(P5) If B is a subset of the set A such that

$$0 \in B \text{ and } x^+ \in B \text{ for all } x \in B,$$

then we have $B = A$.

(b) Let A be a set fulfilling the axioms of Peano. Then the set A is called a **Peano set**.

2.2 Theorem. *There exists exactly one Peano set ω such that*

$$0 := \emptyset \text{ and } A^+ := A \cup \{A\} \text{ for all } A \in \omega.$$

Proof. In Unit *Successor Sets and the Axioms of Peano* [Garden 2020c] the minimal successor set ω is defined, and it is shown that the set ω is a Peano set with the desired properties. □

Definition of the Natural Numbers:

2.3 Definition. Let ω be the Peano set such that

$$0 := \emptyset \text{ and } A^+ := A \cup \{A\} \text{ for all } A \in \omega \text{ (Theorem 2.2).}$$

The set ω is called the **set of the natural numbers** and is denoted by $\mathbb{N}_0$.

French / German. Set of the natural numbers = Ensemble des entiers naturels = Menge der natürlichen Zahlen.

2.4 Definition. (a) We set

$$\begin{array}{rcl} 0 & := & \emptyset \text{ (empty set)} \\ 1 & := & 0^+ = 0 \cup \{0\} = \{0\} = \{\emptyset\} \\ 2 & := & 1^+ = 1 \cup \{1\} = \{0,1\} = \big\{\emptyset, \{\emptyset\}\big\} \\ 3 & := & 2^+ = 2 \cup \{2\} = \{0,1,2\} = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\} \\ & \ldots & \\ 1437 & := & 1436^+ = 1436 \cup \{1436\} = \{0, 1, \ldots, 1436\} \\ & \ldots & \end{array}$$

using the decimal number system without further explanation. The above defined numbers are called **natural numbers**.

(b) For each natural number n, we set $n + 1 := n^+$.

(c) We denote by $\mathbb{N} := \mathbb{N}_0 \setminus \{0\}$ the set of the natural numbers without the number 0.

French / German. Natural number = Entier naturel = Natürliche Zahl.

Note that the number 0 is defined to be a natural number. Some authors do define the number 0 to be natural, others don't. We follow Bourbaki.

Main Properties of the Natural Numbers:

We summarize the previous results in the following theorem:

2.5 Theorem. *(a) The set $\mathbb{N}_0$ contains the element $0 := \emptyset$.*

(b) For each natural number n, the set $n + 1 := n \cup \{n\}$ is a natural number.

(c) We have $n + 1 \neq 0$ for all natural numbers n.

(d) Let n and m be two natural numbers such that $n + 1 = m + 1$. Then we have $n = m$.

(e) Let M be a subset of the set $\mathbb{N}_0$ such that

$$0 \in M \text{ and } m + 1 \in M \text{ for all } m \in M.$$

Then we have $M = \mathbb{N}_0$.

(f) Let M be a subset of the set $\mathbb{N}$ such that

$$0 \notin M, 1 \in M \text{ and } m + 1 \in M \text{ for all } m \in M.$$

Then we have $M = \mathbb{N}$.

(g) Let n be a natural number. If $n \neq 0$, then there exists a natural number k such that $n = k + 1$.

(h) The mapping $\alpha : \mathbb{N}_0 \to \mathbb{N}$, $n \mapsto n + 1$ is bijective.

Proof. Note that we have $\mathbb{N}_0 = \omega$ and that the set $\mathbb{N}_0 = \omega$ is a Peano set (Theorem 2.2).

(a) The assertion follows from Axiom (P1) of Definition 2.1.

(b) The assertion follows from Axiom (P2) of Definition 2.1.

(c) The assertion follows from Axiom (P3) of Definition 2.1.

(d) The assertion follows from Axiom (P4) of Definition 2.1.

(e) The assertion follows from Axiom (P5) of Definition 2.1.

(f) Let $M' := M \cup \{0\}$. Then the set M' is a set such that

$$0 \in M' \text{ and } m' + 1 \in M' \text{ for all } m' \in M'.$$

It follows from (e) that $M' = \mathbb{N}_0$ implying that $M = \mathbb{N}$.

(g) Let

$$Z := \{0\} \cup \{x \in \mathbb{N} \mid \exists\, y \in \mathbb{N} \text{ such that } x = y + 1\}.$$

By definition of the set Z, it contains the element 0. If x is an element of the set Z, then the element $x + 1$ is obviously contained in the set Z. It follows from Axiom (P5) of Definition 2.1 that $Z = \mathbb{N}_0$.

(h) The assertion follows from (c), (d) and (g). □

For the proof of Proposition 2.8 we need the following property of transitive sets which is explained in detail in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c].

2.6 Definition. A set A is called a **transitive** set if every element of the set A is at the same time a subset of the set A.

2.7 Proposition. *(a) The set $\mathbb{N}_0$ of the natural numbers is a transitive set.*

(b) Each natural number n is a transitive set.

Proof. It is shown in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c] that the minimal successor set ω is a transitive set and that each element of the set ω is a transitive set. The assertion follows from the fact that we have defined $\mathbb{N}_0 := \omega$ and that the elements of the set $\mathbb{N}_0$ are exactly the natural numbers. □

2.8 Proposition. *Let m and n be two natural numbers such that the set m is a subset of the set $n + 1$. Then we have*

$$m \subseteq n \text{ or } m = n + 1.$$

Proof. We distinguish the following two cases:

Case 1. *The number n is an element of the set m.*

Since the set m is a transitive set (Proposition 2.7), the number n is also a subset of the set m. It follows that

$$n + 1 = n \cup \{n\} \subseteq m \subseteq n + 1,$$

hence $m = n + 1$.

Case 2. *The number n is no element of the set m.*

Since the set m is a subset of the set $n + 1 = n \cup \{n\}$, it follows that the set m is a subset of the set n. □

Historical Notes:

Historical notes can be found at the end of Section 10.

3 The Principle of Induction and the Recursive Definition of Functions

The Principle of Induction:

3.1 Theorem. (Principle of Induction) *Suppose that $(A_n)_{n \in \mathbb{N}_0}$ is a family of sentences with the following properties:*

(i) $n = 0$: The sentence A_0 is true.

(ii) $n \mapsto n + 1$: If the sentence A_n is true, then the sentence A_{n+1} is also true.

Then every sentence of the family $(A_n)_{n \in \mathbb{N}_0}$ is true.

Proof. Let $\mathcal{A} := \{n \in \mathbb{N}_0 \mid A_n \text{ is true}\}$. It follows from Theorem 2.5 that $\mathcal{A} = \mathbb{N}_0$. □

3.2 Definition. The principle stated in Theorem 3.1 is called the **principle of induction**. It is denoted by:

$n = 0$: Show that the sentence A_0 is true.

$n \mapsto n + 1$: Show that the sentence A_n implies the sentence A_{n+1}.

Then one can conclude that the sentence A_n is true for all natural numbers n.

Note that the principle of induction can also be applied on the set $\mathbb{N}$. In this case, we consider the cases $n = 1$ and $n \mapsto n + 1$.

French / German. Principle of induction = Raisonnement par récurrence = Prinzip der vollständigen Induktion.

3.3 Remark. Note that the principle of induction is not a principle of logical deduction but simply a theorem.

Recursive Definition of Functions:

We will base the possibility of the recursive definition of a function $\alpha : \mathbb{N}_0 \to X$ from the set $\mathbb{N}_0$ of the natural numbers into a set X (Theorem 3.5) on the corresponding result for the minimal successor set ω explained in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c]:

3.4 Theorem. (Recursive Definition) *Let X be a non-empty set, let $f : X \to X$ be a function from the set X into itself, and let a be an element of the set X.*

Then there exists exactly one function $\alpha : \omega \to X$ from the minimal successor set ω into the set X fulfilling the following conditions:

(i) We have $\alpha(\emptyset) = a$.

(ii) We have $\alpha(A^+) = f(\alpha(A))$ for each element A of the set ω.

Proof. See Unit *Successor Sets and the Axioms of Peano* [Garden 2020c]. □

Since $\mathbb{N}_0 = \omega$, we may reformulate the recursive definition of functions (Theorem 3.4) for natural numbers as follows:

3.5 Theorem. (Recursive Definition of a Function) *Let X be a non-empty set, let $f : X \to X$ be a function from the set X into itself, and let a be an element of the set X.*

Then there exists exactly one function $\alpha : \mathbb{N}_0 \to X$ from the set $\mathbb{N}_0$ into the set X fulfilling the following conditions:

(i) We have $\alpha(0) = a$.

(ii) We have $\alpha(n + 1) = f(\alpha(n))$ for each natural number n.

Proof. The proof follows from the definition $\mathbb{N}_0 := \omega$ and Theorem 3.4. □

French / German. Recursive definition of a function = Définition d'une fonction par récurrence = Rekursive Definition einer Funktion.

4 The Addition of Natural Numbers

Definition of the Addition of Natural Numbers:

4.1 Proposition. *Let m be a natural number. Then there exists exactly one function*

$$\alpha_m : \mathbb{N}_0 \to \mathbb{N}_0$$

from the set $\mathbb{N}_0$ into itself such that

$$\alpha_m(0) = m \text{ and } \alpha_m(n + 1) = \alpha_m(n) + 1 \text{ for all } n \in \mathbb{N}_0. \tag{1}$$

Proof. Let $X := \mathbb{N}_0$, and let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined by

$$f(x) := x + 1.$$

By Theorem 3.5, there exists exactly one function $\alpha_m : \mathbb{N}_0 \to X = \mathbb{N}_0$ such that

$$\alpha_m(0) = m \text{ and } \alpha_m(n + 1) = f\big(\alpha_m(n)\big) = \alpha_m(n) + 1.$$

□

4.2 Definition. Let m and n be two natural numbers, and let $\alpha_m : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined in Proposition 4.1. Set

$$m + n := \alpha_m(n). \tag{2}$$

The operation

$$+ : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, (m, n) \mapsto m + n$$

is called the **addition of natural numbers**. The element $m + n$ is called the **sum of the natural numbers** m **and** n.

French / German. Addition of natural numbers = Addition des entiers naturels = Addition natürlicher Zahlen. Sum = Somme = Summe.

4.3 Remark. The definition of the addition of natural numbers in Definition 4.2 relies on the consideration that

$$m + n = m + \underbrace{1 + \ldots + 1}_{n \text{ times}}.$$

An alternative way to define the addition of two natural numbers m and n is to choose two disjoint sets A and B with $|A| = m$ and $|B| = n$ elements and to define

$$m + n := |A \mathbin{\dot\cup} B|.$$

This approach is explained in Unit *Cardinal Arithmetics* [Garden 2020d].

Addition is Associative:

4.4 Proposition. *(a) We have*

$$m + 0 = m \textit{ for all } m \in \mathbb{N}_0. \tag{3}$$

(b) We have

$$m + (n + 1) = (m + n) + 1 \textit{ for all } m, n \in \mathbb{N}_0. \tag{4}$$

Proof. (a) We have

$$m + 0 \overset{(2)}{=} \alpha_m(0) \overset{(1)}{=} m \text{ for all } m \in \mathbb{N}_0.$$

(b) We have

$$m + (n + 1) \overset{(2)}{=} \alpha_m(n + 1) \overset{(1)}{=} \alpha_m(n) + 1 \overset{(2)}{=} (m + n) + 1 \text{ for all } m, n \in \mathbb{N}_0.$$

□

4.5 Theorem. *The addition in the set* $\mathbb{N}_0$ *is associative, that is, we have*

$$(k + m) + n = k + (m + n) \textit{ for all } k, m, n \in \mathbb{N}_0. \tag{5}$$

Proof. Let k, m and n be natural numbers. We proceed by induction on n:

$n = 0$: We have

$$(k + m) + 0 \overset{(3)}{=} k + m \overset{(3)}{=} k + (m + 0).$$

$n \mapsto n+1$: We have

$$\begin{aligned}(k+m)+(n+1) &\overset{(4)}{=} \big((k+m)+n\big)+1 \overset{(\mathrm{Ind.})}{=} \big(k+(m+n)\big)+1 \\ &\overset{(4)}{=} k+\big((m+n)+1\big) \overset{(4)}{=} k+\big(m+(n+1)\big).\end{aligned}$$

□

Addition is Commutative:

4.6 Proposition. *(a) We have*

$$0+n = n \text{ for all } n \in \mathbb{N}_0. \tag{6}$$

(b) We have

$$(m+1)+n = (m+n)+1 \text{ for all } m, n \in \mathbb{N}_0. \tag{7}$$

Proof. (a) Let n be a natural number. We proceed by induction on n:

$n = 0$: We have

$$0+0 \overset{(3)}{=} 0.$$

$n \mapsto n+1$:

$$0+(n+1) \overset{(5)}{=} (0+n)+1 \overset{(\mathrm{Ind.})}{=} n+1.$$

(b) Let m and n be two natural numbers. We proceed by induction on n:

$n = 0$: We have

$$(m+1)+0 \overset{(3)}{=} m+1 \overset{(3)}{=} (m+0)+1.$$

$n \mapsto n+1$: We have

$$\begin{aligned}(m+1)+(n+1) &\overset{(5)}{=} \big((m+1)+n\big)+1 \overset{(\mathrm{Ind.})}{=} \big((m+n)+1\big)+1 \\ &\overset{(5)}{=} \big(m+(n+1)\big)+1.\end{aligned}$$

□

4.7 Theorem. *The addition in the set $\mathbb{N}_0$ is commutative, that is, we have*

$$m+n = n+m \text{ for all } m, n \in \mathbb{N}_0. \tag{8}$$

Proof. Let m and n be two natural numbers. We proceed by induction on n:

$n = 0$: We have

$$m+0 \overset{(3)}{=} m \overset{(6)}{=} 0+m.$$

$n \mapsto n+1$: We have

$$m+(n+1) \overset{(5)}{=} (m+n)+1 \overset{(\mathrm{Ind.})}{=} (n+m)+1 \overset{(7)}{=} (n+1)+m.$$

□

The Additive Cancellation Laws:

4.8 Theorem. *Let x, m and n be three natural numbers.*

(a) If $x + m = x + n$, then we have $m = n$.

(b) If $m + x = n + x$, then we have $m = n$.

(c) If $m + n = 0$, then we have $m = 0$ and $n = 0$.

Proof. (a) We proceed by induction on x:

$x = 0$: If $0 + m = 0 + n$, it follows from (6) that $m = n$.

$x \mapsto x + 1$: Suppose that $(x + 1) + m = (x + 1) + n$. It follows from (7) that

$$(x + m) + 1 = (x + n) + 1.$$

By Theorem 2.5, it follows that $x + m = x + n$. By induction, it follows that $m = n$.

(b) If $m + x = n + x$, then it follows from (8) that $x + m = x + n$. By (a), we get $m = n$.

(c) Case 1. *Assume that* $m \neq 0$.

By Theorem 2.5, there exists a natural number m' such that $m = m' + 1$. It follows that

$$0 = m + n = m' + 1 + n = (m' + n) + 1,$$

in contradiction to Theorem 2.5.

Case 2. *Assume that* $n \neq 0$.

The assertion follows from Case 1 since we have $0 = m + n = n + m$. □

4.9 Theorem. *Let m and n be two natural numbers. Then exactly one of the following cases occurs:*

(i) We have $n = m$.

(ii) There exists a natural number $k \neq 0$ such that $n = m + k$.

(iii) There exists a natural number $k \neq 0$ such that $m = n + k$.

Proof. *Existence:* Let m and n be two natural numbers. We proceed by induction on n:

$n = 0$: If $m = 0$, then we have $m = 0 = n$, that is Case (i).

If $m \neq 0$, then we have

$$m = 0 + m = n + m,$$

that is, Case (iii) with $k = m$.

$n \mapsto n + 1$: By induction, the numbers n and m fulfill one of the Conditions (i), (ii) or (iii).

Case 1. Suppose that $m = n$. Then we have $n + 1 = m + 1$, that is, Case (ii) with $k = 1$.

Case 2. Suppose that there exists a natural number $r \neq 0$ such that $n = m + r$. It follows that

$$n + 1 = m + (r + 1),$$

that is, Case (ii) with $k = r + 1$.

Case 3. Suppose that there exists a natural number $r \neq 0$ such that $m = n + r$.

If $r = 1$, then we have $n + 1 = m$, that is, Case (i).

If $r \neq 1$, there exists a natural number $k \neq 0$ such that $r = k + 1$. It follows that

$$m = n + r = n + k + 1 = n + 1 + k, \text{ that is, Case (iii).}$$

Uniqueness: Assume that the cases (i) and (ii) occur simultaneously. Then we have $n = m$ and $n = m + k$ for a natural number $k \neq 0$. It follows that $m = m + k$ implying that $k = 0$, a contradiction.

Assume that the cases (i) and (iii) occur simultaneously. Then we have $n = m$ and $m = n + k$ for a natural number $k \neq 0$. It follows that $n = n + k$ implying that $k = 0$, a contradiction.

Assume that the cases (ii) and (iii) occur simultaneously. Then we have $n = m + r$ and $m = n + s$ for two natural numbers $r \neq 0$ and $s \neq 0$. It follows that

$$m = n + s = (m + r) + s = m + (r + s)$$

implying that $r + s = 0$ implying that $r = 0$ and $s = 0$, a contradiction. □

5 The Multiplication of Natural Numbers

Definition of the Multiplication of Natural Numbers:

5.1 Proposition. *Let m be a natural number. Then there exists exactly one function*

$$\beta_m : \mathbb{N}_0 \to \mathbb{N}_0$$

from the set $\mathbb{N}_0$ into itself such that

$$\beta_m(0) = 0 \text{ and } \beta_m(n+1) = \beta_m(n) + m \text{ for all } n \in \mathbb{N}_0. \tag{9}$$

Proof. Let $X := \mathbb{N}_0$ and let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined by

$$f(x) := x + m.$$

By Theorem 3.5, there exists exactly one function $\beta_m : \mathbb{N}_0 \to X = \mathbb{N}_0$ such that

$$\beta_m(0) = m \text{ and } \beta_m(n+1) = f\big(\beta_m(n)\big) = \beta_m(n) + m.$$

□

5.2 Definition. Let m and n be two natural numbers, and let $\beta_m : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined in Proposition 5.1. Set

$$m \cdot n := \beta_m(n). \tag{10}$$

The operation

$$\cdot : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, (m, n) \mapsto m \cdot n$$

is called the **multiplication of natural numbers**. The element $m \cdot n$ is called the **product of the natural numbers m and n**. We often write mn instead of $m \cdot n$.

French / German. Multiplication of natural numbers = Multiplication des entiers naturels = Multiplikation natürlicher Zahlen. Product = Produit = Produkt.

5.3 Remark. The definition of the multiplication of natural numbers in Definition 5.2 relies on the consideration that

$$mn = \underbrace{m + \ldots + m}_{n \text{ times}}.$$

An alternative way to define the multiplication of two natural numbers m and n is to choose two sets A and B with $|A| = m$ and $|B| = n$ elements and to define

$$m \cdot n := |A \times B|.$$

This approach is explained in Unit *Cardinal Arithmetics* [Garden 2020d].

Elementary Properties of the Multiplication:

5.4 Proposition. *(a) We have*

$$m \cdot 0 = 0 \textit{ for all } m \in \mathbb{N}_0. \tag{11}$$

(b) We have

$$m \cdot 1 = m \textit{ for all } m \in \mathbb{N}_0. \tag{12}$$

(c) We have

$$m(n+1) = mn + m \textit{ for all } m, n \in \mathbb{N}_0. \tag{13}$$

Proof. (a) We have

$$m \cdot 0 \overset{(10)}{=} \beta_m(0) \overset{(9)}{=} 0 \text{ for all } m \in \mathbb{N}_0.$$

(b) We have

$$m \cdot 1 \overset{(10)}{=} \beta_m(1) \overset{(9)}{=} \beta_m(0+1) \overset{(9)}{=} \beta_m(0) + m = 0 + m = m \text{ for all } m \in \mathbb{N}_0.$$

(c) We have

$$m(n+1) \overset{(10)}{=} \beta_m(n+1) \overset{(9)}{=} \beta_m(n) + m \overset{(10)}{=} mn + m \text{ for all } m, n \in \mathbb{N}_0.$$

□

5.5 Proposition. *We have*

$$k(m+n) = km + kn \textit{ for all } k, m, n \in \mathbb{N}_0. \tag{14}$$

Proof. Let k, m and n be three natural numbers. We proceed by induction on n:

$n = 0$: We have

$$k(m+0) = km = km + 0 \overset{(11)}{=} km + k \cdot 0.$$

$n \mapsto n+1$: We have

$$\begin{aligned} k\big(m + (n+1)\big) &= k\big((m+n)+1\big) \overset{(13)}{=} k(m+n) + k \\ &\overset{(\text{Ind.})}{=} km + kn + k \overset{(13)}{=} km + k(n+1). \end{aligned}$$

□

Multiplication is Associative:

5.6 Theorem. *The multiplication in the set* $\mathbb{N}_0$ *is associative, that is, we have*

$$(km)n = k(mn) \text{ for all } k, m, n \in \mathbb{N}_0. \tag{15}$$

Proof. Let k, m and n be three natural numbers. We proceed by induction on n:

$n = 0$: We have

$$(km) \cdot 0 \overset{(11)}{=} 0 \overset{(11)}{=} k \cdot 0 \overset{(11)}{=} k(m \cdot 0).$$

$n \mapsto n+1$: We have

$$\begin{aligned} (km)(n+1) &\overset{(13)}{=} (km)n + km \overset{(\mathrm{Ind.})}{=} k(mn) + km \\ &\overset{(14)}{=} k\big((mn)+m\big) \overset{(13)}{=} k\big(m(n+1)\big). \end{aligned}$$

□

5.7 Proposition. *(a) We have*

$$0 \cdot n = 0 \text{ for all } n \in \mathbb{N}_0. \tag{16}$$

(b) We have

$$(m+1)n = mn + n \text{ for all } m, n \in \mathbb{N}_0. \tag{17}$$

Proof. (a) We proceed by induction on n:

$n = 0$:

$$0 \cdot 0 \overset{(11)}{=} 0.$$

$n \mapsto n+1$: Note that we have

$$0 \cdot 1 \overset{(10)}{=} \beta_0(1) = \beta_0(0+1) \overset{(9)}{=} \beta_0(0) + 0 \overset{(9)}{=} 0 + 0 = 0.$$

It follows that

$$0 \cdot (n+1) \overset{(13)}{=} 0 \cdot n + 0 \cdot 1 \overset{(\mathrm{Ind.})}{=} 0 + 0 = 0.$$

(b) We proceed by induction on n:

$n = 0$: We have

$$(m+1) \cdot 0 \overset{(11)}{=} 0 = 0 + 0 \overset{(11)}{=} m \cdot 0 + 0.$$

$n \mapsto n+1$: We have

$$\begin{aligned} (m+1)(n+1) &\overset{(14)}{=} (m+1)n + (m+1) \overset{(\mathrm{Ind.})}{=} mn + n + m + 1 \\ &= (mn+m) + n + 1 \overset{(13)}{=} m(n+1) + (n+1). \end{aligned}$$

□

Multiplication is Commutative:

5.8 Theorem. *The multiplication in the set* $\mathbb{N}_0$ *is commutative, that is, we have*

$$mn = nm \text{ for all } m, n \in \mathbb{N}_0. \tag{18}$$

Proof. Let m and be two natural numbers. We proceed by induction on n:

$n = 0$: We have

$$m \cdot 0 \overset{(11)}{=} 0 \overset{(16)}{=} 0 \cdot m.$$

$n \mapsto n+1$: We have

$$m(n+1) \overset{(13)}{=} mn + m \overset{(\text{Ind.})}{=} nm + m \overset{(17)}{=} (n+1)m.$$

□

The Distributive Laws:

5.9 Theorem. *The distributive laws hold in the set* $\mathbb{N}_0$*, that is:*

(a) We have

$$k(m+n) = km + kn \text{ for all } k, m, n \in \mathbb{N}_0. \tag{19}$$

(b) We have

$$(k+m)n = kn + mn \text{ for all } k, m, n \in \mathbb{N}_0. \tag{20}$$

Proof. (a) follows from Proposition 5.5.

(b) Let k, m and n be three natural numbers. Then we have

$$(k+m)n \overset{(18)}{=} n(k+m) \overset{(a)}{=} nk + nm \overset{(18)}{=} kn + mn.$$

□

The Multiplicative Cancellation Laws:

5.10 Theorem. *Let x, m and n be three natural numbers.*

(a) If $mn = 0$, then we have $m = 0$ or $n = 0$.

(b) If $mn = 1$, then we have $m = 1$ and $n = 1$.

(c) If $xm = xn$, then we have $x = 0$ or $m = n$.

(d) If $mx = nx$, then we have $x = 0$ or $m = n$.

Proof. (a) Let $mn = 0$, and suppose that $n \neq 0$. We have to show that $m = 0$. Since $n \neq 0$, there exists a natural number n' such that $n = n' + 1$. It follows that

$$0 = mn = m(n' + 1) = mn' + m.$$

By Theorem 4.8, we have $m = 0$ (and $mn' = 0$).

(b) Let $mn = 1$. We have $n \neq 0$ and $m \neq 0$ since $m \cdot 0 = 0 \cdot n = 0$.

Since $m \neq 0$ and $n \neq 0$, there exist two natural numbers m' and n' such that $m = m' + 1$ and $n = n' + 1$. It follows from Theorem 5.9 that

$$1 = mn = (m'+1)(n'+1) = m'n' + n' + m' + 1.$$

By Theorem 4.8, we have $0 = m'n' + m' + n'$. Again, by Theorem 4.8, we have $m' = 0$, $n' = 0$ (and $m'n' = 0$). It follows from $m' = 0$ and from $n' = 0$ that

$$m = m' + 1 = 0 + 1 = 1 \text{ and } n = n' + 1 = 0 + 1 = 1.$$

(c) Let $xm = xn$, and suppose that $x \neq 0$. Assume that $m \neq n$. In view of Theorem 4.9, there exists a natural number $0 \neq r$ such that $n = m + r$ or $m = n + r$.

Case 1. Suppose that $n = m + r$. Then we have

$$xm = xn = x(m + r) = xm + xr.$$

It follows from Theorem 4.8 that $xr = 0$. By (a), we have $x = 0$ or $r = 0$. Since $x \neq 0$, we have $r = 0$, implying that $m = n$, in contradiction to the assumption that $m \neq n$.

Case 2. Suppose that $m = n+r$. The proof is analog to the proof of Case 1 using the equation $xn = xm = x(n + r) = xn + xr$.

(d) Suppose that $mx = nx$. Then we have $xm = xn$, and it follows from part (c) that $x = 0$ or $m = n$.

6 The Power of Natural Numbers

Definition of the n^{th} power:

6.1 Proposition. *Let m be a natural number. Then there exists exactly one function*

$$\gamma_m : \mathbb{N}_0 \to \mathbb{N}_0$$

from the set $\mathbb{N}_0$ into itself such that

$$\gamma_m(0) = 1 \text{ and } \gamma_m(n + 1) = \gamma_m(n) \cdot m \text{ for all } n \in \mathbb{N}_0. \tag{21}$$

Proof. Let $X := \mathbb{N}_0$ and let $f : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined by

$$f(x) := x \cdot m.$$

By Theorem 3.5, there exists exactly one function $\gamma_m : \mathbb{N}_0 \to X = \mathbb{N}_0$ such that

$$\gamma_m(0) = 1 \text{ and } \gamma_m(n + 1) = f\big(\gamma_m(n)\big) = \gamma_m(n) \cdot m.$$

□

6.2 Definition. Let m and n be two natural numbers, and let $\gamma_m : \mathbb{N}_0 \to \mathbb{N}_0$ be the function from the set $\mathbb{N}_0$ into itself defined in Proposition 6.1. Set

$$m^n := \gamma_m(n). \tag{22}$$

The value m^n is called the n^{th} **power of the number** m. The operation $(m, n) \mapsto m^n$ is called **exponentiation**.

French / German. Power of a natural number = Puissance d'un entier naturel = Potenz einer natürlichen Zahl. Exponentiation = Exponentiation = Potenzieren (= Exponentiation).

6.3 Remark. The definition of the exponentiation of natural numbers in Definition 6.2 relies on the consideration that

$$m^n = \underbrace{m \cdot \ldots \cdot m}_{n \text{ times}}.$$

An alternative way to define the exponentiation m^n for two natural numbers m and n is to choose two sets A and B with $|A| = m$ and $|B| = n$ elements and to define

$$m^n := |\{\alpha : B \to A \mid \alpha \text{ is a function from the set } B \text{ into the set } A\}|.$$

This approach is explained in Unit *Cardinal Arithmetics* [Garden 2020d].

Elementary Properties of the n^{th} power:

6.4 Proposition. *(a) We have*

$$m^0 = 1 \text{ for all } m \in \mathbb{N}_0. \tag{23}$$

(b) We have

$$m^1 = m \text{ for all } m \in \mathbb{N}_0. \tag{24}$$

(c) We have

$$m^{n+1} = m^n \cdot m \text{ for all } m, n \in \mathbb{N}_0. \tag{25}$$

Proof. (a) Let m be a natural number. Then we have $m^0 \overset{(22)}{=} \gamma_m(0) \overset{(21)}{=} 1$.

(b) Let m be a natural number. Then we have

$$m^1 \overset{(22)}{=} \gamma_m(1) = \gamma_m(0+1) \overset{(21)}{=} \gamma_m(0) \cdot m \overset{(21)}{=} 1 \cdot m = m.$$

(c) Let m and n be two natural numbers. Then we have

$$m^{n+1} \overset{(22)}{=} \gamma_m(n+1) \overset{(21)}{=} \gamma_m(n) \cdot m \overset{(22)}{=} m^n \cdot m.$$

□

6.5 Theorem. *(a) We have*

$$(km)^n = k^n \cdot m^n \text{ for all } k, m, n \in \mathbb{N}_0. \tag{26}$$

(b) We have

$$k^{m+n} = k^m \cdot k^n \text{ for all } k, m, n \in \mathbb{N}_0. \tag{27}$$

(c) We have

$$(k^m)^n = k^{mn} \text{ for all } k, m, n \in \mathbb{N}_0. \tag{28}$$

Proof. (a) Let k, m and n be three natural numbers. We proceed by induction on n:

$n = 0$: We have

$$(km)^0 \overset{(23)}{=} 1 = 1 \cdot 1 \overset{(23)}{=} k^0 \cdot m^0.$$

$n \mapsto n+1$: We have

$$(km)^{n+1} \overset{(25)}{=} (km)^n \cdot km \overset{(\text{Ind.})}{=} k^n \cdot m^n \cdot km = k^n \cdot k \cdot m^n \cdot m \overset{(25)}{=} k^{n+1} \cdot m^{n+1}.$$

(b) Let k, m and n be three natural numbers. We proceed by induction on n:

$n = 0$: We have

$$k^{m+0} = k^m = k^m \cdot 1 \overset{(23)}{=} k^m \cdot k^0.$$

$n \mapsto n+1$: We have

$$k^{m+(n+1)} = k^{(m+n)+1} \overset{(25)}{=} k^{m+n} \cdot k \overset{(\text{Ind.})}{=} k^m \cdot k^n \cdot k \overset{(25)}{=} k^m \cdot k^{n+1}.$$

(c) Let k, m and n be three natural numbers. We proceed by induction on n:

$n = 0$: We have

$$(k^m)^0 \overset{(23)}{=} 1 \overset{(23)}{=} k^0 = k^{m \cdot 0}.$$

$n \mapsto n+1$: We have

$$(k^m)^{n+1} \overset{(25)}{=} (k^m)^n \cdot k^m \overset{(\text{Ind.})}{=} k^{mn} \cdot k^m \overset{(27)}{=} k^{mn+m} = k^{m(n+1)}.$$

□

6.6 Proposition. *Let x, m and n be three natural numbers.*

(a) If $x^m = 0$, then we have $m \neq 0$ and $x = 0$.

(b) If $x^m = 1$, then we have $m = 0$ or $x = 1$.

(c) If $x^m = x^n$, then we have $x = 0$ or $x = 1$ or $m = n$.

Proof. (a) Suppose that $x^m = 0$. Assume that $m = 0$. Then we have $x^m = 1$, a contradiction. In order to show that $x = 0$, we proceed by induction on m:

$m = 1$: Suppose that $x^1 = 0$. Then we have $x = x^1 = 0$.

$m \mapsto m+1$: Suppose that $x^{m+1} = 0$. By Theorem 6.5, we have

$$0 = x^{m+1} = x^m \cdot x.$$

It follows from Theorem 5.10 that $x = 0$ or $x^m = 0$. If $x^m = 0$, it follows by induction that $x = 0$.

(b) Suppose that $x^m = 1$ and $m \neq 0$. We have to show that $x = 1$.

Since $m \neq 0$, there exists a natural number m' such that $m = m' + 1$. It follows that

$$1 = x^m = x^{m'+1} = x^{m'} \cdot x.$$

It follows from Theorem 5.10 that $x = 1$ (and $x^{m'} = 1$).

(c) Suppose that $x^m = x^n$. Assume that $m \neq n$. We have to show that $x = 0$ or $x = 1$. By Theorem 4.9, there exists a natural number $0 \neq r$ such that $m = n + r$ or $n = m + r$. W.l.o.g. suppose that $n = m + r$. It follows that

$$x^m \cdot 1 = x^m = x^n = x^{m+r} = x^m \cdot x^r.$$

It follows from Theorem 5.10 that $x^m = 0$ or $x^r = 1$.

If $x^m = 0$, it follows from (a) that $x = 0$.

If $x^r = 1$, then it follows from (b) that $r = 0$ or $x = 1$.

If $r = 0$, then we have $m = n$, a contradiction. □

6.7 Remark. Note that it will follow from Proposition 8.17 that the equation $x^n = y^n$ implies that $n = 0$ or $x = y$.

7 Factorial of n and the Fibonacci Numbers

For the recursive definition of some functions the process described in Theorem 3.5 has to be extended. We will present two examples, the factorial $n!$ of a natural number n (see Example 7.1) and the Fibonacci numbers (see Example 7.2).

7.1 Example. The factorial $n!$ of a natural number n is defined as follows:

$$0! := 1, \ n! := 1 \cdot 2 \cdot \ldots \cdot n \text{ for all } n \in \mathbb{N}$$

or, equivalently, by

$$0! := 1, \ (n+1)! := n!(n+1) \text{ for all } n \in \mathbb{N}.$$

Hence, we have to construct a function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ with the property that

$$\beta(0) = 1 \text{ and } \beta(n+1) = \beta(n) \cdot (n+1) \text{ for all } n \in \mathbb{N}_0.$$

The idea is to construct a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that

$$\alpha(n) = (n!, n+1) \text{ for all } n \in \mathbb{N}_0$$

and to define the function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ by $\beta := \mathrm{pr}_1 \circ \alpha$ where

$$\mathrm{pr}_1 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0, \ \mathrm{pr}_1 : (x, y) \mapsto x$$

denotes the projection on the first component (for more details about projections see Unit *Families and the Axiom of Choice* [Garden 2020a]).
For, let $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be the function defined by

$$f(m, n) := (mn, m+1) \text{ for all } m, n \in \mathbb{N}_0.$$

By Theorem 3.5, there exists a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that

$$\alpha(0) = (1, 1) \text{ and } \alpha(n+1) = f\big(\alpha(n)\big) \text{ for all } n \in \mathbb{N}_0.$$

Let $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\beta := \mathrm{pr}_1 \circ \alpha$. As indicated above we have to verify that

$$\beta(0) = 1 \text{ and } \beta(n+1) = \beta(n) \cdot (n+1) \text{ for all } n \in \mathbb{N}_0.$$

For, we will show by induction on n that

$$\beta(0) = 1, \ \alpha(n) = \big(\beta(n), n+1\big) \text{ and } \beta(n+1) = \beta(n) \cdot (n+1) \text{ for all } n \in \mathbb{N}_0.$$

$n = 0$: We have $\alpha(0) = (1, 1)$ implying that $\beta(0) = \mathrm{pr}_1\big(\alpha(0)\big) = 1$. In particular, we have

$$\alpha(0) = (1, 1) = \big(\beta(0), 0+1\big).$$

$n \mapsto n+1$: By induction, we have

$$\alpha(n+1) = f\big(\alpha(n)\big) = f\big(\beta(n), n+1\big) = \big(\beta(n)(n+1), n+2\big)$$

implying that $\beta(n+1) = \mathrm{pr}_1\big(\alpha(n+1)\big) = \beta(n)(n+1)$. In particular, we have

$$\alpha(n+1) = \big(\beta(n)(n+1), n+2\big) = \big(\beta(n+1), (n+1)+1\big).$$

The uniqueness of the function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ follows by induction.

7.2 Example. The Fibonacci numbers $\big(F_n\big)_{n\in\mathbb{N}_0}$ are defined as follows:

$$F_0 := 1,\ F_1 := 1,\ F_{n+2} := F_n + F_{n+1} \text{ for all } n \in \mathbb{N}_0.$$

Hence, we have to construct a function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ with the property that

$$\beta(0) = 1,\ \beta(1) = 1 \text{ and } \beta(n+2) = \beta(n+1) + \beta(n) \text{ for all } n \in \mathbb{N}_0.$$

The idea is to construct a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that

$$\alpha(0) = (1,0) \text{ and } \alpha(n+1) = (F_{n+1}, F_n) \text{ for all } n \in \mathbb{N}_0$$

and to define the function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ by $\beta := \mathrm{pr}_1 \circ \alpha$ where

$$\mathrm{pr}_1 : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0,\ \mathrm{pr}_1 : (x,y) \mapsto x$$

denotes the projection on the first component as in Example 7.1.

For, let $f : \mathbb{N}_0 \times \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ be the function defined by

$$f(m,n) := (m+n, m) \text{ for all } m, n \in \mathbb{N}_0.$$

By Theorem 3.5, there exists a function $\alpha : \mathbb{N}_0 \to \mathbb{N}_0 \times \mathbb{N}_0$ such that

$$\alpha(0) = (1,0) \text{ and } \alpha(n+1) = f\big(\alpha(n)\big) \text{ for all } n \in \mathbb{N}_0.$$

Let $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\beta := \mathrm{pr}_1 \circ \alpha$. As indicated above we have to verify that

$$\beta(0) = 1,\ \beta(1) = 1 \text{ and } \beta(n+2) = \beta(n+1) + \beta(n) \text{ for all } n \in \mathbb{N}_0.$$

For, we will show by induction on n that $\alpha(0) = (1,0)$, $\alpha(1) = (1,1)$, $\beta(0) = 1$, $\beta(1) = 1$ and

$$\begin{aligned} \alpha(n+2) &= \big(\beta(n+2), \beta(n+1)\big) \text{ and} \\ \beta(n+2) &= \beta(n+1) + \beta(n) \text{ for all } n \in \mathbb{N}_0. \end{aligned}$$

$n = 0$: We have $\alpha(0) = (1,0)$ implying that $\beta(0) = \mathrm{pr}_1\big(\alpha(0)\big) = 1$.

$n = 1$: We have

$$\alpha(1) = f\big(\alpha(0)\big) = f(1,0) = (1,1)$$

implying that $\beta(1) = \mathrm{pr}_1\big(\alpha(1)\big) = 1$. In particular, we have

$$\alpha(1) = (1,1) = \big(\beta(1), \beta(0)\big).$$

$n+1 \mapsto n+2$: By induction, we have

$$\alpha(n+2) = f\big(\alpha(n+1)\big) = f\big(\beta(n+1), \beta(n)\big) = \big(\beta(n+1) + \beta(n), \beta(n+1)\big)$$

implying that $\beta(n+2) = \mathrm{pr}_1\big(\alpha(n+2)\big) = \beta(n+1) + \beta(n)$. In particular, we have

$$\alpha(n+2) = \big(\beta(n+1) + \beta(n), \beta(n+1)\big) = \big(\beta(n+2), \beta(n+1)\big).$$

The uniqueness of the function $\beta : \mathbb{N}_0 \to \mathbb{N}_0$ follows by induction.

8 The Standard Order on the Natural Numbers

Definition of the Standard Order on the Set $\mathbb{N}_0$:

8.1 Definition. Let m and n be two natural numbers. We set $m \leqslant n$ if and only if there exists a natural number r such that

$$n = m + r.$$

The order $\leqslant$ is called the **standard order** on the set of the natural numbers.

French / German. Standard order on the natural numbers = Ordre naturel sur les nombres entiers = Standardordnung auf den natürlichen Zahlen.

8.2 Theorem. *The pair $(\mathbb{N}_0, \leqslant)$ is a totally ordered set with respect to the standard order on the natural numbers (Definition 8.1).*

Proof. Since

$$n = n + 0 \text{ for all } n \in \mathbb{N}_0,$$

we have $n \leqslant n$ for all natural numbers n.

Let m and n be two natural numbers such that $m \leqslant n$ and $n \leqslant m$. Then there exist two natural numbers r and s such that

$$n = m + r \text{ and } m = n + s.$$

It follows that $n = m + r = n + s + r$ implying that $r = 0$ and $s = 0$. Hence, we have $m = n$.

Let k, m and n be three natural numbers such that $k \leqslant m$ and $m \leqslant n$. Then there exist two natural numbers r and s such that

$$m = k + r \text{ and } n = m + s.$$

It follows that $n = m + s = k + r + s$ implying that $k \leqslant n$.

Finally, let m and n be two natural numbers. Then it follows from Theorem 4.9 that $m \leqslant n$ or $n \leqslant m$. $\square$

8.3 Proposition. *Let m and n be two natural numbers. Then the following two conditions are equivalent:*

(i) We have $m < n$.

(ii) There exists a natural number $r \neq 0$ such that $n = m + r$.

Proof. (i) $\Rightarrow$ (ii): Suppose that $m < n$. Then it follows that $m \leqslant n$ implying that there exists a natural number r such that $n = m+r$. Assuming $r = 0$, we get $m = n$, a contradiction.

(ii) $\Rightarrow$ (i): Suppose that $n = m + r$ for a natural number $r \neq 0$. Then it follows that $m \leqslant n$. Assuming that $m = n$, we obtain $r = 0$, a contradiction. □

Elementary Properties of the Ordered Set $\mathbb{N}_0$:

8.4 Proposition. *Let m and n be two natural numbers.*

(a) If $n \neq 0$, then we have $0 < n$.

(b) We have $n < n + 1$.

(c) We have $m \leqslant n$ if and only if $m + 1 \leqslant n + 1$.

(d) We have $m < n$ if and only if $m + 1 < n + 1$.

(e) If $m < n$, then we have $m + 1 \leqslant n$.

(f) If $m < n + 1$, then we have $m \leqslant n$.

(g) If $m < n \leqslant m + 1$, then we have $n = m + 1$.

Proof. Let m and n be two natural numbers.

(a) Suppose that $n \neq 0$. By Proposition 8.3, it follows from $n = 0 + n$ and $n \neq 0$ that $0 < n$.

(b) By Proposition 8.3, it follows from $n + 1 = n + 1$ and $1 \neq 0$ that $n < n + 1$.

(c) Suppose that $m \leqslant n$. Then there exists a natural number r such that $n = m+r$. It follows that $n + 1 = m + 1 + r$, hence $m + 1 \leqslant n + 1$.

Suppose that $m+1 \leqslant n+1$. Then there exists a natural number r such that $n+1 = m+1+r$. It follows that $n = m + r$, hence $m \leqslant n$.

(d) Suppose that $m < n$. Then there exists a natural number $r \neq 0$ such that $n = m + r$. It follows that $n + 1 = m + 1 + r$, hence $m + 1 < n + 1$.

Suppose that $m+1 < n+1$. Then there exists a natural number $r \neq 0$ such that $n+1 = m+1+r$. It follows that $n = m + r$, hence $m < n$.

(e) Suppose that $m < n$. Then there exists a natural number $r \neq 0$ such that $n = m+r$. Since $r \neq 0$, there exists a natural number r' such that $r = r' + 1$. It follows that

$$n = m + r = m + r' + 1 = m + 1 + r'$$

implying that $m + 1 \leqslant n$.

(f) Suppose that $m < n+1$. Then there exists a natural number $r \neq 0$ such that $n+1 = m+r$. Since $r \neq 0$, there exists a natural number r' such that $r = r' + 1$. It follows that

$$n + 1 = m + r = m + r' + 1$$

implying that $n = m + r'$. It follows that $m \leqslant n$.

(g) Suppose that $m < n \leqslant m + 1$. It follows from (e) that $m + 1 \leqslant n$ implying that $m + 1 \leqslant n \leqslant m + 1$. It follows that $m + 1 = n$. □

8.5 Theorem. *Let m and n be two natural numbers.*

(a) We have $m \subseteq n$ if and only if $m \leqslant n$.

(b) We have $m \subset n$ if and only if $m < n$.

Proof. (a) Step 1. *Suppose that $m \subseteq n$. Then we have $m \leqslant n$:*

We have to show that there exists a natural number k such that $n = m + k$.

We proceed by induction on n:

$n = 0$: It follows from

$$m \subseteq n = 0 = \emptyset$$

that $m = \emptyset = 0$. We get

$$n = m + k \text{ with } k = 0.$$

$n \mapsto n + 1$: Suppose that $m \subseteq n + 1$. By Proposition 2.8, we have

$$m \subseteq n \text{ or } m = n + 1.$$

If $m \subseteq n$, then it follows by induction that $m \leqslant n \leqslant n + 1$. If $m = n + 1$, then obviously, $m \leqslant n + 1$.

Step 2. *Suppose that $m \leqslant n$. Then we have $m \subseteq n$:*

We proceed by induction on n:

$n = 0$: It follows from $m \leqslant n$ that $m = 0$. Hence, we have $m = 0 = n$ implying that $m \subseteq n$.

$n \mapsto n + 1$: Suppose that we have $m \leqslant n + 1$. Then there exists a natural number k such that $n + 1 = m + k$.

Case 1. If $k = 0$, then we have $n + 1 = m$ implying that $m \subseteq n + 1$.

Case 2. If $k \neq 0$, then there exists a natural number r such that $k = r + 1$. It follows that

$$n + 1 = m + k = m + r + 1.$$

implying that $n = m + r$, that is, $m \leqslant n$. By induction, it follows that $m \subseteq n$, Since $n \subseteq n + 1$, it follows that $m \subseteq n + 1$.

(b) follows from (a) □

8.6 Remark. Theorem 8.5 can also be used for an alternative definition of the standard order on the natural numbers as follows: For two natural numbers m and n, we define

$$m \leqslant n \text{ if and only if } m \subseteq n.$$

8.7 Theorem. *Let n be a natural number. Then we have $n = \{x \in \mathbb{N}_0 \mid x < n\}$.*

Proof. We proceed by induction on n:

$n = 0$: We have $0 = \emptyset = \{x \in \mathbb{N}_0 \mid x < 0\}$.

$n \mapsto n + 1$: Suppose that $n = \{x \in \mathbb{N}_0 \mid x < n\}$ for some natural number n.

Step 1. *We have* $\{x \in \mathbb{N}_0 \mid x < n\} \cup \{n\} \subseteq \{x \in \mathbb{N}_0 \mid x < n+1\}$*:*

If $x < n$, it follows from $n < n+1$ (Proposition 8.4) that $x < n+1$.

If $x = n$, it follows from $n < n+1$ (Proposition 8.4) that $x < n+1$.

Step 2. *We have* $\{x \in \mathbb{N}_0 \mid x < n+1\} \subseteq \{x \in \mathbb{N}_0 \mid x < n\} \cup \{n\}$*:*

If $x < n+1$, then it follows from Proposition 8.4 that we have $x \leqslant n$. It follows that $x < n$ or $x = n$.

Step 3. *We have* $n+1 = \{x \in \mathbb{N}_0 \mid x < n+1\}$*:*

By Step 1 and 2 and by induction, we have

$$n+1 = n \cup \{n\} = \{x \in \mathbb{N}_0 \mid x < n\} \cup \{n\} = \{x \in \mathbb{N}_0 \mid x < n+1\}.$$

□

8.8 Definition. Let m and n be two natural numbers. We set

$$\{m, m+1, \ldots, n\} := \begin{cases} \{x \in \mathbb{N}_0 \mid m \leqslant x \text{ and } x \leqslant n\} & \text{if} \quad m \leqslant n \text{ and} \\ \emptyset & \text{if} \quad m > n. \end{cases}$$

Note that $\{n, n+1, \ldots, n\} = \{n\}$.

8.9 Theorem. *Let n be a natural number. Then we have*

$$n+1 = \{0, 1, \ldots, n\}.$$

Proof. The assertion follows from Theorem 8.7. □

8.10 Theorem. *Let A be a subset of the set $\mathbb{N}_0$. If the set A is non-empty, then it has a minimal element.*

Proof. Suppose that the set A is non-empty. If A contains the element 0, then 0 is the minimal element of the set A. So we may assume that the element 0 is not contained in the set A.

We claim that there exists an element k of the set A such that

$$\{0, 1, \ldots, k\} \cap A = \emptyset \text{ and } k+1 \in A:$$

Assume on the contrary that no such element k exists. Let

$$B := \{x \in \mathbb{N}_0 \mid \{0, 1, \ldots, x\} \cap A = \emptyset\}.$$

It follows that

$$0 \in B \text{ and } x+1 \in B \text{ whenever } x \in B.$$

Hence, we have $B = \mathbb{N}_0$ implying that

$$A \cap \mathbb{N}_0 = A \cap B = \emptyset,$$

a contradiction. It follows that the element $k+1$ is the minimal element of the set A. □

Addition and Order:

8.11 Proposition. *Let* a, b, c *and* d *be four natural numbers.*

(a) If $a \leqslant c$ *and* $b \leqslant d$, *then we have* $a + b \leqslant c + d$.

(b) If $a < c$ *and* $b \leqslant d$, *then we have* $a + b < c + d$.

(c) If $a \leqslant c$ *and* $b < d$, *then we have* $a + b < c + d$.

Proof. (a) It follows from $a \leqslant c$, $b \leqslant d$ and Theorem 8.5 that there exist two natural numbers r and s such that $c = a + r$ and $d = b + s$ implying that

$$c + d = a + r + b + s = (a + b) + (r + s).$$

By Theorem 8.5, we get $a + b \leqslant c + d$.

(b) It follows from $a < c$, $b \leqslant d$ and Theorem 8.5 that there exists a natural number $r \neq 0$ and a natural number s such that $c = a + r$ and $d = b + s$ implying that

$$c + d = a + r + b + s = (a + b) + (r + s).$$

Since $r + s \neq 0$, it follows from Theorem 8.5 that $a + b < c + d$.

(c) It follows from (b) that

$$a + b = b + a < d + c = c + d.$$

□

8.12 Proposition. *Let* x, m *and* n *be three natural numbers.*

(a) If $x + m \leqslant x + n$, *then we have* $m \leqslant n$.

(b) If $x + m < x + n$, *then we have* $m < n$.

Proof. (a) It follows from $x + m \leqslant x + n$ and Theorem 8.5 that there exists a natural number r such that

$$x + m = x + n + r.$$

It follows from Theorem 4.8 that $m = n + r$ implying that $m \leqslant n$.

(b) It follows from (a) that $m \leqslant n$. Assuming $m = n$ we get $x + m = x + n$, a contradiction. □

Multiplication and Order:

8.13 Proposition. *Let* a, b, c *and* d *be four natural numbers.*

(a) If $a \leqslant c$ *and* $b \leqslant d$, *then we have* $ab \leqslant cd$.

(b) If $a < c$, $b \leqslant d$ *and* $d \neq 0$, *then we have* $ab < cd$.

(c) If $a \leqslant c$, $b < d$ *and* $c \neq 0$, *then we have* $ab < cd$.

Proof. (a) It follows from $a \leqslant c$, $b \leqslant d$ and Theorem 8.5 that there exist two natural numbers r and s such that $c = a + r$ and $d = b + s$ implying that

$$cd = (a + r)(b + s) = ab + (as + rb + rs).$$

By Theorem 8.5, we get $ab \leqslant cd$.

(b) It follows from $a < c$, $b \leqslant d$ and Theorem 8.5 that there exists a natural number $r \neq 0$ and a natural number s such that $c = a + r$ and $d = b + s$ implying that

$$cd = (a + r)(b + s) = ab + as + r(b + s) = ab + (as + rd).$$

Since $r \neq 0$ and $d \neq 0$, we have $rd \neq 0$ implying that $as + rd \neq 0$. It follows from Theorem 8.5 that $ab < cd$.

(c) follows from (b). □

8.14 Proposition. *Let x, m and n be three natural numbers.*

(a) If $x \neq 0$ and $xm \leqslant xn$, then we have $m \leqslant n$.

(b) If $x \neq 0$ and $xm < xn$, then we have $m < n$.

Proof. (a) Assume that $n < m$. It follows from Theorem 8.5 that there exists a natural number $r \neq 0$ such that $m = n + r$ implying that

$$xm = xn + xr.$$

Since $x \neq 0$ and $r \neq 0$, we have $xr \neq 0$. It follows from Theorem 8.5 that $xn < xm$, a contradiction.

(b) It follows from (a) that $m \leqslant n$. Assume that $m = n$. Then we have $xm = xn$, a contradiction. □

Exponentiation and Order:

8.15 Proposition. *Let x, y, m and n be four natural numbers.*

(a) If $x \leqslant y$, then we have $x^n \leqslant y^n$.

(b) If $n \neq 0$ and $x < y$, then we have $x^n < y^n$.

(c) If $x \neq 0$ and $m \leqslant n$, then we have $x^m \leqslant x^n$.

(d) If $x \neq 0$, $x \neq 1$ and $m < n$, then we have $x^m < x^n$.

(e) If $x \neq 0$, $x \leqslant y$ and $m \leqslant n$, then we have $x^m \leqslant y^n$.

(f) If $x \neq 0$ $x \neq 1$, $x < y$ and $m \leqslant n$, then we have $x^m < y^n$.

(g) If $x \neq 0$ $x \neq 1$, $x \leqslant y$ and $m < n$, then we have $x^m < y^n$.

Proof. (a) Let $x \leqslant y$. We proceed by induction on n:

$n = 0$: We have $x^0 = 1 \leqslant 1 = y^0$.

$n \mapsto n + 1$: By induction, we have $x^n \leqslant y^n$. It follows from Proposition 8.13 that

$$x^{n+1} = x^n x \leqslant y^n y = y^{n+1}.$$

(b) Let $n \neq 0$ and $x < y$. We proceed by induction on n:

$n = 1$: We have $x^1 = x < y = y^1$.

$n \mapsto n + 1$: By induction, we have $x^n < y^n$. It follows from Proposition 8.13 that

$$x^{n+1} = x^n x < y^n y = y^{n+1}.$$

(c) Let $x \neq 0$ and $m \leqslant n$. Since $m \leqslant n$, there exists a natural number r such that $n = m + r$. Since $x \neq 0$, we have $1 \leqslant x^r$. It follows from Proposition 8.13 that

$$x^m \leqslant x^m x^r = x^{m+r} = x^n.$$

(d) Let $x \neq 0$, $x \neq 1$ and $m < n$. Since $m < n$, there exists a natural number $r \neq 0$ such that $n = m + r$. Since $x > 1$ and $r \neq 0$, we have $1 < x^r$. It follows from Proposition 8.13 that

$$x^m < x^m x^r = x^{m+r} = x^n.$$

(e) Let $x \neq 0$, $x \leqslant y$ and $m \leqslant n$. We have

$$x^m \overset{(a)}{\leqslant} y^m \overset{(c)}{\leqslant} y^n.$$

(f) Let $x \neq 0$ $x \neq 1$, $x < y$ and $m \leqslant n$. We have $x^m \leqslant x^n < y^n$, hence $x^m < y^n$.

(g) $x \neq 0$ $x \neq 1$, $x \leqslant y$ and $m < n$. We have $x^m \leqslant y^m < y^n$, hence $x^m < y^n$. □

8.16 Proposition. *Let x, y, m and n be four natural numbers.*

(a) If $n \neq 0$ and $x^n \leqslant y^n$, then we have $x \leqslant y$.

(b) If $x^n < y^n$, then we have $x < y$.

(c) If $x \neq 0$, $x \neq 1$ and $x^m \leqslant x^n$, then we have $m \leqslant n$.

(d) If $x \neq 0$ and $x^m < x^n$, then we have $m < n$.

Proof. (a) Let $n \neq 0$ and $x^n \leqslant y^n$. Assume that $y < x$. By Proposition 8.15, we have $y^n < x^n$, a contradiction.

(b) Let $x^n < y^n$, Note that $n \neq 0$ since we have $x^n < y^n$, but $x^0 = 1 = y^0$. It follows from (a) that $x \leqslant y$. Assuming $x = y$, we obtain $x^n = y^n$, a contradiction.

(c) Let $x \neq 0$, $x \neq 1$ and $x^m \leqslant x^n$. Assume that $n < m$. By Proposition 8.15, we have $x^n < x^m$, a contradiction.

(d) Let $x \neq 0$ and $x^m < x^n$. Note that $x \neq 1$ since we have $x^m < x^n$, but $1^m = 1 = 1^n$. It follows from (c) that $m \leqslant n$. Assume that $m = n$. Then we have $x^m = x^n$, a contradiction. □

8.17 Proposition. *Let x, y, and n be three natural numbers.*

If $x^n = y^n$, then we have $n = 0$ or $x = y$.

Proof. Let $x^n = y^n$ and suppose that $n \neq 0$. It follows from $x^n = y^n$ that we have

$$x^n \leqslant y^n \text{ and } y^n \leqslant x^n.$$

By Proposition 8.16, we get $x \leqslant y$ and $y \leqslant x$ implying that $x = y$. □

Isomorphism between $\mathbb{N}_0$ and $\mathbb{N}$:

We recall the definition of an isomorphism of ordered sets introduced in Unit *Ordered Sets and the Lemma of Zorn* [Garden 2020b]:

8.18 Definition. Let $A = (A, \leqslant_A)$ and $B = (B, \leqslant_B)$ be two ordered sets. A bijective mapping $\alpha : A \to B$ is called an **isomorphism from the ordered set** A **onto the ordered set** B if we have

$$x \leqslant_A y \text{ if and only if } \alpha(x) \leqslant_B \alpha(y) \text{ for all } x, y \in A.$$

8.19 Proposition. *The function* $\alpha : \mathbb{N}_0 \to \mathbb{N}$, $\alpha : n \mapsto n+1$ *is an isomorphism form the ordered set* $(\mathbb{N}_0, \leqslant)$ *onto the ordered set* $(\mathbb{N}, \leqslant)$*. In particular, the sets* $(\mathbb{N}_0, \leqslant)$ *and* $(\mathbb{N}, \leqslant)$ *are isomorphic.*

Proof. By Theorem 2.5, the function $\alpha : \mathbb{N}_0 \to \mathbb{N}$ is bijective. By Proposition 8.4, we have

$$n \leqslant m \Leftrightarrow n+1 \leqslant m+1 \Leftrightarrow \alpha(n) \leqslant \alpha(m).$$

□

9 Generalized Arithmetic Laws

Sums and Products:

9.1 Definition. Let m and n be two natural numbers, let $I := \{m, m+1, \ldots, n\}$ (Definition 8.8), and let x_j be a natural number for each element j of the set I.

(a) If $m > n$, we set

$$\sum_{j=m}^{n} x_j := 0 \text{ (empty sum).}$$

(b) If $m = n$, we set

$$\sum_{j=m}^{m} x_j := x_m.$$

(c) If $m < n$, we have $n > 0$, and there exists a natural number n' such that $n = n' + 1$. It follows that $m \leqslant n'$. We set

$$\sum_{j=m}^{n} x_j := \left(\sum_{j=m}^{n'} x_j \right) + x_n \text{ (recursive definition).}$$

(d) We set

$$x_k + x_{k+1} + \ldots + x_n := \sum_{j=k}^{n} x_j.$$

9.2 Definition. Let m and n be two natural numbers, let $I := \{m, m+1, \ldots, n\}$ (Definition 8.8), and let x_j be a natural number for each element j of the set I.

(a) If $m > n$, we set

$$\prod_{j=m}^{n} x_j := 1 \text{ (empty product).}$$

(b) If $m = n$, we set

$$\prod_{j=m}^{m} x_j := x_m.$$

(c) If $m < n$, we have $n > 0$, and there exists a natural number n' such that $n = n' + 1$. It follows that $m \leqslant n'$. We set

$$\prod_{j=m}^{n} x_j := \left(\prod_{j=m}^{n'} x_j\right) \cdot x_n \text{ (recursive definition)}.$$

(d) We set

$$x_k \cdot x_{k+1} \cdot \ldots \cdot x_n := \prod_{j=k}^{n} x_j.$$

Generalized Associative Law:

9.3 Proposition. *Let k, m and n be three natural numbers such that $k \leqslant m < n$, and let*

$$x_k, x_{k+1}, \ldots, x_m, x_{m+1}, \ldots, x_n$$

be a sequence of natural numbers.

(a) We have

$$\Big(\sum_{j=k}^{m} x_j\Big) + \Big(\sum_{j=m+1}^{n} x_j\Big) = \sum_{j=k}^{n} x_j.$$

(b) We have

$$\Big(\prod_{j=k}^{m} x_j\Big) \cdot \Big(\prod_{j=m+1}^{n} x_j\Big) = \prod_{j=k}^{n} x_j.$$

Proof. (a) Since we have $m < n$, there exists a natural number $r \geqslant 1$ such that $n = m + r$. We have to show that

$$\Big(\sum_{j=k}^{m} x_j\Big) + \Big(\sum_{j=1}^{r} x_{m+j}\Big) = \sum_{j=k}^{n} x_j :$$

We proceed by induction on r:

$r = 1$: Note that $n = m + r = m + 1$. By Definition 9.1, we have

$$\sum_{j=k}^{m+1} x_j = \Big(\sum_{j=k}^{m} x_j\Big) + x_{m+1}.$$

$r \mapsto r+1$: We have

$$\begin{aligned}
&\Big(\sum_{j=k}^{m} x_j\Big) + \Big(\sum_{j=1}^{r+1} x_{m+j}\Big) \\
&= \Big(\sum_{j=k}^{m} x_j\Big) + \left(\Big(\sum_{j=1}^{r} x_{m+j}\Big) + x_{m+r+1}\right) \text{ (Definition 9.1)} \\
&= \left(\Big(\sum_{j=k}^{m} x_j\Big) + \Big(\sum_{j=1}^{r} x_{m+j}\Big)\right) + x_{m+r+1} \text{ (associative law)} \\
&= \Big(\sum_{j=k}^{n} x_j\Big) + x_{n+1} \text{ (induction and } n = m + r) \\
&= \sum_{j=k}^{n+1} x_j \text{ (Definition 9.1).}
\end{aligned}$$

(b) The proof of (b) is identical with the proof of (a). We only have to replace the signs $+$ and $\sum$ by the signs $\cdot$ and $\prod$ and to use Definition 9.2 instead of Definition 9.1. □

9.4 Proposition. (Generalized Associative Law) *Let k and n be two natural numbers with $k < n$, and let $x_{k+1}, \ldots, x_n$ be a sequence of natural numbers. In addition, let $r \geqslant 1$ be a natural number, and let $n_0, n_1, \ldots, n_r$ be a sequence of natural numbers such that*

$$k = n_0 < n_1 < \ldots < n_r = n.$$

Let r' be the natural number such that $r = r' + 1$.

(a) We have

$$\sum_{i=0}^{r'} \Big(\sum_{j=n_i+1}^{n_{i+1}} x_j \Big) = \sum_{j=k+1}^{n} x_j.$$

(b) We have

$$\prod_{i=0}^{r'} \Big(\prod_{j=n_i+1}^{n_{i+1}} x_j \Big) = \prod_{j=k+1}^{n} x_j.$$

Proof. (a) We proceed by induction on r:

$r = 1$: Note that we have $r' = 0$, $k = n_0$ and $n = n_r = n_1$. It follows that

$$\sum_{i=0}^{r'} \Big(\sum_{j=n_i+1}^{n_{i+1}} x_j \Big) = \sum_{i=0}^{0} \Big(\sum_{j=n_i+1}^{n_{i+1}} x_j \Big) = \sum_{j=n_0+1}^{n_1} x_j = \sum_{j=k+1}^{n} x_j.$$

$r \mapsto r+1$: Let k and n be two natural numbers with $k < n$, and let $x_{k+1}, \ldots, x_n$ be a sequence of natural numbers. Let $r \geqslant 1$ be a natural number, and let $n_0, n_1, \ldots, n_r, n_{r+1}$ be natural numbers such that

$$k = n_0 < n_1 < \ldots n_r < n_{r+1} = n.$$

We have to show that

$$\sum_{i=0}^{r} \Big(\sum_{j=n_i+1}^{n_{i+1}} x_j \Big) = \sum_{j=k+1}^{n} x_j :$$

Note that we have $r = r' + 1$, $k = n_0$ and $n = n_{r+1}$. It follows that

$$\begin{aligned}
&\sum_{i=0}^{r}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big) \\
&= \sum_{i=0}^{r'+1}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big) = \left(\sum_{i=0}^{r'}\Big(\sum_{j=n_i+1}^{n_{i+1}} x_j\Big)\right) + \Big(\sum_{j=n_r+1}^{n_{r+1}} x_j\Big) \text{ (Definition 9.1)} \\
&= \Big(\sum_{j=k+1}^{n_r} x_j\Big) + \Big(\sum_{j=n_r+1}^{n_{r+1}} x_j\Big) \text{ (induction and } r = r'+1) \\
&= \sum_{j=k+1}^{n_{r+1}} x_j = \sum_{j=k+1}^{n} x_j \text{ (Proposition 9.3 and } n = n_{r+1}).
\end{aligned}$$

(b) The proof of (b) is identical with the proof of (a). We just have to replace the signs $+$ and $\sum$ by the signs $\cdot$ and $\prod$. □

Generalized Commutative Law:

9.5 Proposition. *Let y be a natural number, and let $x_1, \ldots, x_n$ be a sequence of natural numbers for some natural number $n \geqslant 1$.*

(a) We have

$$y + x_1 + \ldots + x_n = x_1 + \ldots + x_n + y.$$

(b) We have

$$y \cdot x_1 \cdot \ldots \cdot x_n = x_1 \cdot \ldots \cdot x_n \cdot y.$$

Proof. (a) We proceed by induction on n:

$n = 1$: By Theorem 4.7, we have $y + x_1 = x_1 + y$.

$n \mapsto n+1$: Let $x_1, \ldots, x_{n+1}$ be a sequence of natural numbers. Then we have

$$\begin{aligned}
y + x_1 + \ldots x_{n+1} &= (y + x_1 + \ldots + x_n) + x_{n+1} \\
&= (x_1 + \ldots + x_n + y) + x_{n+1} \text{ (induction)} \\
&= (x_1 + \ldots + x_n) + (y + x_{n+1}) \text{ (Proposition 9.4)} \\
&= (x_1 + \ldots + x_n) + (x_{n+1} + y) \text{ (Theorem 4.7)} \\
&= x_1 + \ldots + x_n + x_{n+1} + y.
\end{aligned}$$

(b) The proof of (b) is identical with the proof of (a). We just have to replace the sign $+$ by the sign $\cdot$ and Theorem 4.7 by Theorem 5.8. □

9.6 Proposition. (Generalized Commutative Law) *Let $x_1, \ldots, x_n$ be a sequence of natural numbers for some natural number $n \geqslant 1$, and let*

$$\alpha : \{1, \ldots, n\} \to \{1, \ldots, n\}$$

be a bijective mapping from the set $\{1, \ldots, n\}$ onto itself.

(a) We have

$$x_{\alpha(1)} + \ldots + x_{\alpha(n)} = x_1 + \ldots + x_n.$$

(b) We have

$$x_{\alpha(1)} \cdot \ldots \cdot x_{\alpha(n)} = x_1 \cdot \ldots \cdot x_n.$$

(c) Let $y_1, \ldots, y_n$ *be a sequence of natural numbers. Then we have*

$$\sum_{i=1}^{n} (x_i + y_i) = \Big(\sum_{i=1}^{n} x_i\Big) + \Big(\sum_{i=1}^{n} y_i\Big) \text{ and } \prod_{i=1}^{n} (x_i y_i) = \Big(\prod_{i=1}^{n} x_i\Big) \cdot \Big(\prod_{i=1}^{n} y_i\Big)$$

Proof. (a) We proceed by induction on n:

$n = 1$: For $n = 1$ we have $\alpha(1) = 1$ implying that $x_{\alpha(1)} = x_1$.

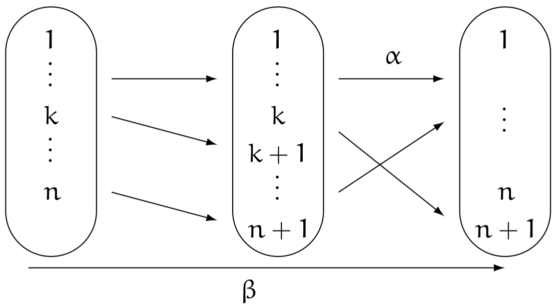
$n \mapsto n+1$: Let $x_1, \ldots, x_{n+1}$ be a sequence of natural numbers, and let

$$\alpha : \{1, \ldots, n+1\} \to \{1, \ldots, n+1\}$$

be a bijective mapping from the set $\{1, \ldots, n+1\}$ onto itself. Then there exists an element k of the set $\{1, \ldots, n+1\}$ such that $\alpha(k) = n+1$.

Define the mapping $\beta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ by

$$\beta(j) := \begin{cases} \alpha(j) & \text{if} \quad 1 \leqslant j < k \\ \alpha(j+1) & \text{if} \quad k \leqslant j \leqslant n. \end{cases}$$



It is easy to verify that the mapping $\beta : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is bijective.

If $k = 1$, then we have

$$\begin{aligned} x_{\alpha(1)} + \ldots + x_{\alpha(n+1)} &= x_{\alpha(k)} + x_{\alpha(2)} + \ldots + x_{\alpha(n+1)} \text{ } (k = 1) \\ &= x_{\alpha(2)} + \ldots + x_{\alpha(n+1)} + x_{\alpha(k)} \text{ (Proposition 9.5)} \\ &= x_{\beta(1)} + \ldots + x_{\beta(n)} + x_{n+1} \text{ (Definition of } \beta \text{ and } \alpha(k) = n+1) \\ &= x_1 + \ldots + x_n + x_{n+1} \text{ (induction).} \end{aligned}$$

If $k > 1$, let k' be the natural number such that $k = k' + 1$. Then we have

$$\begin{aligned} & x_{\alpha(1)} + \ldots + x_{\alpha(n+1)} \\ &= x_{\alpha(1)} + \ldots + x_{\alpha(k')} + x_{\alpha(k)} + x_{\alpha(k+1)} \ldots + x_{\alpha(n+1)} \\ &= x_{\alpha(1)} + \ldots + x_{\alpha(k')} + x_{\alpha(k+1)} \ldots + x_{\alpha(n+1)} + x_{\alpha(k)} \text{ (Proposition 9.5)} \\ &= x_{\beta(1)} + \ldots + x_{\beta(k')} + x_{\beta(k)} + \ldots + x_{\beta(n)} + x_{n+1} \text{ (Definition of } \beta \text{ and } \alpha(k) = n+1) \\ &= x_1 + \ldots + x_n + x_{n+1} \text{ (induction).} \end{aligned}$$

(b) follows as in (a).

(c) follows from (a) and (b). □

Generalized Distributive Law:

9.7 Proposition. *Let x be a natural number, and let $y_1, \ldots, y_n$ be a sequence of natural numbers for some natural number $n \geqslant 1$. Then we have*

$$\sum_{j=1}^{n}(xy_j) = x\Big(\sum_{j=1}^{n} y_j\Big) = \Big(\sum_{j=1}^{n} y_j\Big)x.$$

Proof. We will prove the first equation by induction on n.

$n = 1$: We have

$$\sum_{j=1}^{n}(xy_j) = \sum_{j=1}^{1}(xy_j) = xy_1 = x\Big(\sum_{j=1}^{1} y_j\Big) = x\Big(\sum_{j=1}^{n} y_j\Big).$$

$n \mapsto n+1$: Let $y_1, \ldots, y_n, y_{n+1}$ be a sequence of natural numbers. Then we have

$$\begin{aligned}
\sum_{j=1}^{n+1}(xy_j) &= \sum_{j=1}^{n}(xy_j) + xy_{n+1} \text{ (Definition 9.1)} \\
&= x\Big(\sum_{j=1}^{n} y_j\Big) + xy_{n+1} \text{ (induction)} \\
&= x\left(\Big(\sum_{j=1}^{n} y_j\Big) + y_{n+1}\right) \text{ (Theorem 5.9)} \\
&= x\Big(\sum_{j=1}^{n+1} y_j\Big) \text{ (Definition 9.1).}
\end{aligned}$$

The second equation follows from the first equation and the fact that the multiplication in $\mathbb{N}_0$ is commutative. □

9.8 Proposition. (Generalized Distributive Law) *Let $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ be two sequences of natural numbers for some natural numbers $m \geqslant 1$ and $n \geqslant 1$. Then we have*

$$(x_1 + \ldots + x_m)(y_1 + \ldots + y_n) = \sum_{i=1}^{m}\sum_{j=1}^{n} x_i y_j.$$

Proof. We proceed by induction on n:

$n = 1$: We have

$$\begin{aligned}
(x_1 + \ldots + x_m)y_1 &= \Big(\sum_{i=1}^{m} x_i\Big)y_1 \text{ (Definition 9.1)} \\
&= \sum_{i=1}^{m}(x_i y_1) \text{ (Proposition 9.7)} \\
&= \sum_{i=1}^{m}\Big(x_i\Big(\sum_{j=1}^{1} y_j\Big)\Big) \text{ (Definition 9.1)} \\
&= \sum_{i=1}^{m}\Big(\sum_{j=1}^{1}(x_i y_j)\Big) \text{ (Proposition 9.7)}
\end{aligned}$$

$n \mapsto n+1$: Let $y_1, \ldots, y_{n+1}$ be a sequence of natural numbers. Then we have

$$
\begin{aligned}
&(x_1 + \ldots + x_m)(y_1 + \ldots + y_n + y_{n+1}) \\
&= (x_1 + \ldots + x_m)(y_1 + \ldots + y_n) + (x_1 + \ldots + x_m)y_{n+1} \text{ (Theorem 5.9)} \\
&= \left(\sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_i y_j\Big)\right) + \Big(\sum_{i=1}^{m} x_i\Big) y_{n+1} \text{ (induction)} \\
&= \sum_{i=1}^{m}\Big(\sum_{j=1}^{n} x_i y_j + x_i y_{n+1}\Big) \text{ (Proposition 9.6)} \\
&= \sum_{i=1}^{m}\Big(x_i\Big(\sum_{j=1}^{n} y_j + y_{n+1}\Big)\Big) \text{ (Proposition 9.7 and Theorem 5.9)} \\
&= \sum_{i=1}^{m} x_i\Big(\sum_{j=1}^{n+1} y_j\Big) \text{ (Definition 9.1)} \\
&= \sum_{i=1}^{m}\sum_{j=1}^{n+1} x_i y_j \text{ (Proposition 9.7)}
\end{aligned}
$$

□

10 Dedekind's Construction of the Natural Numbers

Richard Dedekind published in 1872 the paper *Stetigkeit und irrationale Zahlen* (Continuity and irrational numbers) [Dedekind 1872] where he gave an axiomatic foundation of the real numbers based on what is today called the *cuts of Dedekind*. For more details see Unit *The Real Numbers* [Garden 2020e].

In 1888 he published the paper *Was sind und was sollen die Zahlen?* (What are numbers and what should they be?) [Dedekind 1888]. In this article he gives a formal definition of *finite* and *infinite* sets and an axiomatic foundation of the natural numbers.

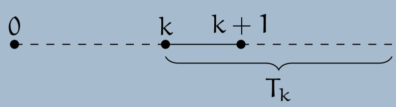
We will explain his brilliant ideas in this section. The ideas of Dedekind are based on his definition of a chain. We start by explaining chains in the context of the natural numbers:

Chains in the Set $\mathbb{N}_0$:

10.1 Definition. Let k be a natural number. The set

$$T_k := \{x \in \mathbb{N}_0 \mid x \geqslant k\}$$

is called a **chain** or, a **Dedekind chain in the set** $\mathbb{N}_0$. In addition, the set $T_\infty := \emptyset$ is called the **empty chain**.



10.2 Proposition. *Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varphi(n) := n+1$ for all natural numbers n, and let A be a subset of the set $\mathbb{N}_0$.*

Then the set A is a chain in the set $\mathbb{N}_0$ if and only if

$$\varphi(A) \subseteq A.$$

Proof. Step 1. *Suppose that the set A is a chain in the set $\mathbb{N}_0$. Then we have $\varphi(A) \subseteq A$:*

If $A = \emptyset$, then we have $\varphi(A) = \varphi(\emptyset) = \emptyset$.

If $A \neq \emptyset$, then there exists a natural number k such that $A = T_k = \{x \in \mathbb{N}_0 \mid x \geqslant k\}$. It follows that

$$\varphi(A) = \varphi(T_k) = \{\varphi(x) \mid x \geqslant k\} = \{x + 1 \mid x \geqslant k\} = T_{k+1} \subseteq T_k.$$

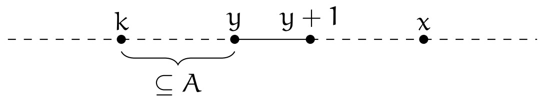
Step 2. *Suppose that we have $\varphi(A) \subseteq A$. Then the set A is a chain in the set $\mathbb{N}_0$:*

If $A = \emptyset$, then the set A is by definition a chain. If $A \neq \emptyset$, then, by Theorem 8.10, there exists a minimal element k in the set A. It follows that

$$A \subseteq \{x \in \mathbb{N}_0 \mid x \geqslant k\} = T_k.$$

Conversely, let x be an element of the set T_k. Assume that the element x is not contained in the set A.

Then there exists an element y of the set A such that the element $y + 1$ is not contained in the set A.



On the other hand we have

$$y + 1 = \varphi(y) \in \varphi(A) \subseteq A,$$

a contradiction. □

10.3 Proposition. *Let $(K_i)_{i \in I}$ be a family of chains in the set $\mathbb{N}_0$. Then the set $\bigcap_{i \in I} K_i$ is also a chain in the set $\mathbb{N}_0$.*

Proof. Let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varphi(n) := n + 1$ for all natural numbers n. It follows from Proposition 10.2 that

$$\varphi(K_i) \subseteq K_i \text{ for all } i \in I.$$

It follows that $\varphi\left(\bigcap_{i \in I} K_i\right) \subseteq \bigcap_{i \in I} K_i$. Again by Proposition 10.2, it follows that the set $\bigcap_{i \in I} K_i$ is a chain in the set $\mathbb{N}_0$. □

10.4 Definition. Let A be a subset of the set $\mathbb{N}_0$, and let

$$\bar{A} := \bigcap \{K \subseteq \mathbb{N}_0 \mid K \text{ is a chain and } A \subseteq K\}.$$

The set $\bar{A}$ is called the **chain in the set $\mathbb{N}_0$ generated by the set A**.

10.5 Proposition. *Let A be a subset of the set $\mathbb{N}_0$, and let $\bar{A}$ be the chain in the set $\mathbb{N}_0$ generated by the set A.*

(a) If $A = \emptyset$, then we have $\bar{A} = A = \emptyset$.

(b) If $A \neq \emptyset$ and if k is the minimal element of the set A, then we have

$$\bar{A} = T_k = \{x \in \mathbb{N}_0 \mid x \geqslant k\}.$$

(c) Let a be a natural number. Then we have

$$\overline{\{a\}} = T_a = \{x \in \mathbb{N}_0 \mid x \geqslant a\}.$$

Proof. (a) is obvious.

(b) Since the set T_k is a chain containing the set A as a subset, we have

$$\bar{A} = \bigcap\{K \subseteq \mathbb{N}_0 \mid K \text{ is a chain and } A \subseteq K\} \subseteq T_k.$$

On the other hand, since the element k belongs to the set A, the set T_k is a subset of the set $\bar{A}$.

(c) follows from (b). □

10.6 Proposition. *Let $A := \{0\}$, and let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varphi(n) := n + 1$ for all natural numbers n. Then we have $\bar{A} = \mathbb{N}_0$ and $\varphi(\bar{A}) = \mathbb{N}$.*

Proof. The assertion follows from Proposition 10.5 and Theorem 2.5. □

Dedekind's Construction of the Natural Numbers:

Dedekind's idea is as follows. He starts with the definition that a set A is called **infinite** if there exists a bijective mapping $\alpha : A \to A'$ from the set A onto a proper subset A' of the set A. As a next step he postulates the existence of infinite sets (see Definition 10.7 and Axiom 10.8).

10.7 Definition. Let A be a set. The set A is called **infinite** if there exists a bijective mapping $\alpha : A \to A'$ from the set A onto a proper subset A' of the set A.

Otherwise, the set A is called **finite**.

10.8 Axiom. (Axiom of Infinity) There exists at least one infinite set.

His next observation is that each infinite set contains a set "similar" to the set of the natural numbers. Today we would say that each infinite set contains a Peano set. In Proposition 10.6 we have seen the following:

Let $A := \{0\}$, and let $\varphi : \mathbb{N}_0 \to \mathbb{N}_0$ be defined by $\varphi(n) := n + 1$ for all natural numbers n. Then we have $\bar{A} = \mathbb{N}_0$ and $\varphi(\bar{A}) = \mathbb{N}$.

Dedekind is now going in the opposite direction: Given an infinite set S he starts with a mapping $\varphi : S \to S$, and he looks for a subset N of the set S which may serve as the set of the natural numbers or, in other words, which is a Peano set. This set N should fulfill the following requirements:

(i) The mapping $\varphi : S \to S$ shall induce a mapping $\alpha : N \to N$. In other words, we need that the set $\varphi(N)$ is a subset of the set N. Later on, we will define

$$x + 1 := \alpha(x) \text{ for all } x \in N.$$

(ii) The set N contains a distinguished element a such that

$$\alpha(x) \neq a \text{ for all } x \in N.$$

Later on, he will set $0 := a$.

(iii) The set N is not too big, in other words, it should contain the elements

$$0, 1 = \alpha(0), 2 = \alpha(1), \ldots,$$

but no further elements.

Proposition 10.6 motivates the following idea: Let $\varphi : S \longrightarrow S$ be a mapping from the set S into itself, let a be an (appropriate) element of the set S, and let N be the chain generated by the set $\{a\}$.

To do so, we have first to generalize the concept of a chain to an arbitrary set S. This is done in Definition 10.9. As a next step we have to define what a chain generated by a subset is. The definition is quite obvious and is the content of Definition 10.12.

10.9 Definition. Let A be a set, and let $\varphi : A \longrightarrow A$ be a mapping from the set A into itself. A subset K of the set A is called a **chain with respect to the mapping** $\varphi : A \longrightarrow A$ if we have

$$\varphi(K) \subseteq K.$$

If no confusion may arise, we also speak of a **chain** instead of a chain with respect to the mapping $\varphi : A \longrightarrow A$.

10.10 Proposition. *Let A be a set, let $\varphi : A \longrightarrow A$ be a mapping from the set A into itself, and let $(K_i)_{i \in I}$ be a (non-empty) family of chains with respect to the mapping $\varphi : A \longrightarrow A$. Then the set*

$$\bigcap_{i \in I} K_i$$

is also a chain.

Proof. If $K = \emptyset$, then we have $\varphi(K) = \varphi(\emptyset) = \emptyset$, and it follows that the set K is a chain.

Let x be an element of the set K. Then the element x is contained in each set K_i. Since each of the sets K_i is a chain, it follows that the element $\varphi(x)$ is contained in each set K_i implying that

$$\varphi(x) \in \bigcap_{i \in I} K_i = K.$$

Hence, the set K is a chain. □

10.11 Proposition. *Let S be a set, let $\varphi : S \longrightarrow S$ be a mapping from the set S into itself, and let A be a subset of the set S.*

(a) The set

$$\bar{A} := \bigcap \{K \subseteq S \mid K \text{ is a chain and } A \subseteq K\}$$

is a chain.

(b) We have

$$A \subseteq \bar{A} \text{ and } A = \bar{A} \text{ if and only if the set } A \text{ is a chain.}$$

(c) Let B be a chain containing the set A as a subset. Then the chain $\bar{A}$ is also a subset

of the set B.

Proof. (a) follows from Proposition 10.10. Note that the set S is a chain containing the set A as a subset. So we do not have an empty intersection.

(b) is obvious.

(c) We have

$$\bar{A} = \bigcap \{K \subseteq S \mid K \text{ is a chain and } A \subseteq K\} \subseteq B.$$

$\square$

10.12 Definition. Let S be a set, let $\varphi : S \longrightarrow S$ be a mapping from the set S into itself, and let A be a subset of the set S.

The set

$$\bar{A} := \bigcap \{K \subseteq S \mid K \text{ is a chain and } A \subseteq K\}$$

defined in Proposition 10.11 is called the **chain generated by the set** A.

The definition of a chain in Definition 10.9 implies that we have

$$\varphi(N) \subseteq N \text{ for } N := \overline{\{a\}}.$$

Hence, the mapping $\varphi : S \longrightarrow S$ induces a mapping $\alpha : N \longrightarrow N$. It remains to guarantee that the mapping $\alpha : N \longrightarrow N$ is injective and that

$$\alpha(x) \neq a \text{ for all } x \in N.$$

The first property can easily be guaranteed if the mapping $\varphi : S \longrightarrow S$ is injective. The second property can be guaranteed if there already exists an element a of the set S such that

$$\varphi(x) \neq a \text{ for all } x \in S.$$

To do so, Dedekind comes back to his definition of an infinite set S stating that there exists a bijective mapping $\varphi : S \longrightarrow S'$ from the set S onto a proper subset S' of the set S. This mapping is by definition injective, and for an element a of the set $S \setminus S'$ we have

$$\varphi(x) \neq a \text{ for all } x \in S.$$

In Theorem 10.13 we will see that this setting works quite well, that is, that the resulting set N with the distinguished element a is a Peano set.

10.13 Theorem. *Let S be an infinite set, let $\varphi : S \longrightarrow S'$ be a bijective mapping from the set S onto a proper subset S' of the set S, let a be an element of the set $S \setminus S'$, and let $N := \overline{\{a\}}$ be the chain generated by the set $\{a\}$.*

(a) The mapping $\varphi : S \longrightarrow S'$ induces an injective mapping $\alpha : N \longrightarrow N$ from the set N into itself.

(b) Set $0 := a$ and

$$x^+ := \alpha(x) \text{ for all } x \in N.$$

Then the set N is a Peano set.

Proof. (a) Since the set N is a chain, we have

$$\varphi(N) \subseteq N$$

implying that the mapping $\varphi : S \to S'$ induces a mapping $\alpha : N \to N$ from the set N into itself. Since the mapping $\varphi : S \to S'$ is bijective, the mapping $\alpha : N \to N$ is injective.

(b) We have to verify the axioms of Peano, that is, Conditions (P1) to (P5) of Definition 2.1:

(P1) *The set* N *contains a distinguished element* 0*:*

By definition of the set N, it contains the element $a = 0$.

(P2) *There is a function* $+ : N \to N$*,* $x \mapsto x^+$ *from the set* N *into itself:*

The assertion follows from the fact that the mapping $\alpha : N \to N$ is a mapping from the set N into itself and from the definition

$$x^+ := \alpha(x) \text{ for all } x \in N.$$

(P3) *We have* $x^+ \neq 0$ *for all elements* x *of the set* N*:*

Since the element a belongs to the set $S \setminus S'$, we have $\varphi(x) \neq a$ for all elements x of the set S. It follows that $\alpha(x) = \varphi(x) \neq a$ for all elements x of the set N. Hence, we have

$$x^+ = \alpha(x) \neq a = 0 \text{ for all } x \in N.$$

(P4) *If* x *and* y *are two elements of the set* N *such that* $x^+ = y^+$*, then we have* $x = y$*:*

The assertion follows from the fact that the mapping $\alpha : N \to N$ is injective.

(P5) *If* B *is a subset of the set* N *such that*

$$0 \in B \textit{ and } x^+ \in B \textit{ for all } x \in B,$$

then we have $B = N$*:*

Since the set B is a subset of the set N, it is also a subset of the set S. We have

$$\varphi(x) = \alpha(x) = x^+ \in B \text{ for all } x \in B$$

implying that

$$\varphi(B) \subseteq B,$$

that is, the set B is a chain with respect to the mapping $\varphi : S \to S$. Since the set B contains the element $0 = a$, it follows from Proposition 10.11 that the set $N = \overline{\{a\}}$ is a subset of the set B. It follows from

$$B \subseteq N \subseteq B$$

that $B = N$. □

Historical Notes:

For the formal definition of finite and infinite sets, for the definition of the Peano sets and for the recursive definition of functions see the historical notes in Unit *Successor Sets and the Axioms of Peano* [Garden 2020c].

The first formal definition of the natural numbers is due to Dedekind as explained in detail above:

71. Erklärung. *Ein System* N *heißt* **einfach unendlich**, *wenn es eine solche ähnliche Abbildung* φ *von* N *in sich selbst gbt, dass* N *als Kette [...] eines Elements erscheint, welches nicht in* $\varphi(N)$ *enthalten ist. Wir nennen dieses Element, das wir im Folgenden durch das Symbol* 1 *bezeichnen wollen, das* **Grundelement** *von* N *und sagen zugleich, das einfach unendliche System* N *sei durch diese Abbildung* φ *geordnet.*

Behalten wir die früheren bequemen Bezeichnungen für die Bilder und Ketten bei [...], so besteht mithin das Wesen eines einfach uendlichen Systems N *in der Existenz einer Abbildung* $\varphi(N)$ *und eines Elements* 1*, die den folgenden Bedingungen* α*,* β*,* γ *und* δ *genügen:*

α. $N' \subseteq N$.

β. $N = 1_0$.

γ. *Das Element* 1 *ist nicht in* N' *enthalten.*

δ. *Die Abbildung* φ *ist ähnlich.*

See [Dedekind 1932, Volume 3, p. 359].

71. Explanation. *A system* N *is called* **simply infinite** *if there exists a similar mapping* φ *of* N *into itself such that* N *is a chain of an element which is not contained in* $\varphi(N)$*. We call this element, which we want to denote by the symbol* 1 *in the following, the* **basic element** *of* N *and we say at the same time that the simply infinite system* N *is ordered by this mapping* φ*.*

If we keep the previous convenient names for the images and chains, [...] the essence of a simply infinite system N *is the existence of a map* $\varphi(N)$ *and an element* 1 *that fulfill the following conditions* α*,* β*,* γ *and* δ*:*

α. $N' \subseteq N$.

β. $N = 1_0$.

γ. *The element* 1 *is not contained in* N'*.*

δ. *The mapping* φ *is similar.*

(Translation by the author.)

"Similar" means injective. "N is a chain of an element" means that the set N is the chain generated by the set $\{a\}$ (see Definition 10.12). Note that Dedekind starts his construction of the set of the natural numbers with the element 1, whereas we started with the element 0.

A simply infinite set is an infinite set N with the property that there exists an element a (called 1) of the set N and an injective mapping $\varphi : N \to N \setminus \{a\}$ such that $N = \overline{\{a\}}$.

Dedekind's condition (α) $(N' \subseteq N)$ means that the set $\varphi(N)$ is a subset of the set N. More generally, Dedekind's notation X' means $\varphi(X)$.

Condition (β) $(N = 1_0)$ means that the set N is the chain generated by the set $\{1\}$. More generally, Dedekind's notation X_0 means $\bar{X}$ (see Definition 10.12).

Condition (γ) (*The element* 1 *is not contained in* N') means that $\varphi(x) \neq 0$ (or $\varphi(x) \neq 1$ if we start the natural numbers with the number 1) for all elements x of the set N.

Finally, Condition (δ) (*The mapping* φ *is similar*) means that the mapping $\varphi : N \to N$ is injective.

*

As a next step Dedekind proves the principle of induction:

80. Satz der vollständigen Induktion. *(Schluss von $\mathfrak{n}$ auf $\mathfrak{n}'$). Um zu beweisen, dass ein Satz für alle Zahlen $\mathfrak{n}$ einer Kette $\mathfrak{m}_0$ gilt, genügt es zu zeigen,*

ρ. *dass er für $\mathfrak{n} = \mathfrak{m}$ gilt und*

σ. *dass aus der Gültigkeit des Satzes für eine Zahl $\mathfrak{n}$ der Kette $\mathfrak{m}_0$ stets seine Gültigkeit auch für die folgende Zahl $\mathfrak{n}'$ folgt.*

See [Dedekind 1932, Volume 3, p. 361].

80. Theorem of induction. *(Deduction from $\mathfrak{n}$ to $\mathfrak{n}'$). To prove that a sentence is valid for all numbers $\mathfrak{n}$ in a chain $\mathfrak{m}_0$, it is sufficient to show*

ρ. *that it holds for $\mathfrak{n} = \mathfrak{m}$ and*

σ. *that the validity of the sentence for a number $\mathfrak{n}$ of the chain $\mathfrak{m}_0$ always implies its validity for the following number $\mathfrak{n}'$.*

(Translation by the author.)

Note that Dedekind's notations $\mathfrak{m}_0$ and $\mathfrak{n}'$ mean

$$\mathfrak{m}_0 = T_{\mathfrak{m}} = \{x \in \mathbb{N}_0 \mid x \geqslant \mathfrak{m}\} \text{ and } \mathfrak{n}' = \mathfrak{n} + 1.$$

*

Dedekind also introduces the standard order on the natural numbers. As we have seen in Proposition 10.5 we have

$$\overline{\{a\}} = T_a = \{x \in \mathbb{N}_0 \mid x \geqslant a\} \text{ for all } a \in \mathbb{N}_0.$$

Since Dedekind uses the notation a_0 for $\overline{\{a\}}$, he introduces the standard order on the set of the natural numbers as follows:

89. Erklärung. *Die Zahl $\mathfrak{m}$ heißt* **kleiner** *als die Zahl $\mathfrak{n}$ [...], wenn die Bedingung*

$$\mathfrak{n}_0 \subseteq \mathfrak{m}_0'$$

erfüllt ist [...].

See [Dedekind 1932, Volume 3, p. 363].

89. Explanation. *The number $\mathfrak{m}$ is called* **smaller** *than the number $\mathfrak{n}$ [...] if the condition*

$$\mathfrak{n}_0 \subseteq \mathfrak{m}_0'$$

is satisfied [...].

(Translation by the author.)

This is equivalent to say that

$$\mathfrak{m} \leqslant \mathfrak{n} \text{ if and only if } \mathfrak{n}_0 = T_{\mathfrak{n}} \subseteq T_{\mathfrak{m}} = \mathfrak{m}_0.$$

Note that we have

$$\mathbb{N}_0 = \{x \in \mathbb{N}_0 \mid x < \mathfrak{n}\} \mathbin{\dot\cup} \{x \in \mathbb{N}_0 \mid x \geqslant \mathfrak{n}\} = \mathfrak{n} \mathbin{\dot\cup} T_{\mathfrak{n}}.$$

So the definition

$$\mathfrak{m} \leqslant \mathfrak{n} \text{ if and only if } T_{\mathfrak{n}} \subseteq T_{\mathfrak{m}}$$

is equivalent to the definition

$$m \leqslant n \text{ if and only if } m \subseteq n.$$

explained in Theorem 8.5 and Remark 8.6. In other words, our definition of the natural numbers is equivalent to say that

$$n = \{x \in \mathbb{N}_0 \mid x < n\},$$

whereas Dedekind's definition corresponds to

$$n = \{x \in \mathbb{N}_0 \mid x \geqslant n\}.$$

Both approaches have their specific advantages.

Finally, Dedekind also introduces the addition, the multiplication and the exponentiation of the natural numbers. All this is done quite similar to the way explained in the previous sections. In fact, all results of this unit are already contained in Dedekind's paper of 1888.

11 Notes and References

A crucial step in the development of the natural numbers was the introduction of our decimal number system which has been invented about 500 AD in Northern India. Arabian mathematicians, in particular Abu Dschafar Muhammad ibn Musa al-Chwarizmi, took up these results and spread them around the world. An excellent account about the history of number systems is the book *Histoire universelle des chiffres* from Georges Ifrah [Ifrah 1981] (for an English and a German translation see [Ifrah 1998] and [Ifrah 1989]).

12 Literature

A list of text books about set theory can be found at Literature about Set Theory. A list of text books about numbers can be found at Literature about Numbers.

Dedekind, Richard (1872). *Stetigkeit und irrationale Zahlen.* Braunschweig: Vieweg (cit. on p. 37).

— (1888). *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg (cit. on pp. 7, 37).

— (1932). *Gesammelte mathematische Werke*. Ed. by Robert Fricke, Emmy Noether, and Öystein Ore. Braunschweig: Vieweg. There are three volumes: Volume 1: (1930), Volume 2: (1931), Volume 3: (1932). (Cit. on pp. 43, 44).

Ifrah, Georges (1981). *Histoire Universelle des Chiffres.* Paris: Editions Seghers. For an English and a German version see [Ifrah 1998] and [Ifrah 1989]. (Cit. on p. 45).

— (1989). *Universalgeschichte der Zahlen.* Frankfurt, New York: Campus-Verlag. German translation of [Ifrah 1981]. (Cit. on p. 45).

— (1998). *Universal History of Numbers.* London: Harville Press. English translation of [Ifrah 1981]. (Cit. on p. 45).

Peano, Giuseppe (1889). *Arithmetices Prinicipia - Nova Methodo Exposita.* Romae and Florentiae: Augustae Taurinorum. This book is published under the name Ioseph Peano. (Cit. on p. 3).

13 Publications of the Mathematical Garden

For a complete list of the publications of the mathematical garden please have a look at www.math-garden.com.

Garden, M. (2020a). *Families and the Axiom of Choice*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-families#nst-families-download (cit. on p. 22).

— (2020b). *Ordered Sets and the Lemma of Zorn*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-ordered-sets#nst-ordered-sets-download (cit. on p. 30).

— (2020c). *Successor Sets and the Axioms of Peano*. Version 1.0.0. URL: https://www.math-garden.com/unit/nst-successor-sets#nst-successor-sets-download (cit. on pp. 3, 7, 9–11, 42).

— (2020d). *Cardinal Arithmetic*. Version 1.0.0. In preparation (cit. on pp. 12, 16, 20).

— (2020e). *The Real Numbers*. Version 1.0.0. In preparation (cit. on p. 37).

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If you are willing to share comments and ideas to improve the present unit or hints about further references, we kindly ask you to send a mail to info@math-garden.com or to use the contact form on www.math-garden.com. Contributions are highly appreciated.

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